Solutions to Chapter 1

5. [10] For each of the wave functions in question 4, find $\bar{x}$, Δx, $\bar{p}$, Δp, and check that the uncertainty relationship $(\Delta x)(\Delta p) \ge \frac{1}{2}\hbar$ is satisfied.

(a) [5] $\psi(x) = (A/\pi)^{1/4}\exp\left(iKx - \frac{1}{2}Ax^2\right)$.

(b) [5] $\psi(x) = \sqrt{\alpha}\exp\left(-\alpha|x|\right)$.

We simply work out each case in a straightforward manner. For part (a), we have

$$\bar{x} = \int_{-\infty}^{\infty} x\psi^*(x)\psi(x)\,dx = \sqrt{\frac{A}{\pi}}\int_{-\infty}^{\infty} xe^{-iKx-Ax^2/2}e^{iKx-Ax^2/2}dx = \sqrt{\frac{A}{\pi}}\int_{-\infty}^{\infty} xe^{-Ax^2}dx = 0\,,$$

$$(\Delta x)^2 = \int_{-\infty}^{\infty}(x-\bar{x})^2\psi^*(x)\psi(x)\,dx = \sqrt{\frac{A}{\pi}}\int_{-\infty}^{\infty}x^2e^{-Ax^2}dx = \sqrt{\frac{A}{\pi}}\Gamma\left(\tfrac{3}{2}\right)A^{-3/2} = \sqrt{\frac{A}{\pi}}\frac{\sqrt{\pi}}{2A^{3/2}} = \frac{1}{2A}\,,$$

$$\bar{p} = \frac{\hbar}{\sqrt{\pi A}}\int_{-\infty}^{\infty}k\tilde{\psi}^*(k)\psi(k)\,dk = \frac{\hbar}{\sqrt{\pi A}}\int_{-\infty}^{\infty}ke^{-(k-K)^2/A}dk = \frac{\hbar}{\sqrt{\pi A}}\frac{\hbar}{\sqrt{\pi A}}\int_{-\infty}^{\infty}(k+K)e^{-k^2/A}dk$$

$$= \frac{\hbar}{\sqrt{\pi A}}\left[0 + K\sqrt{\pi A}\right] = \hbar K\,,$$

$$(\Delta p)^2 = \frac{1}{\sqrt{\pi A}}\int_{-\infty}^{\infty}(\hbar k - \hbar K)^2\tilde{\psi}^*(k)\psi(k)\,dk = \frac{\hbar^2}{\sqrt{\pi A}}\int_{-\infty}^{\infty}(k-K)^2e^{-(k-K)^2/A}dk =$$

$$= \frac{\hbar^2}{\sqrt{\pi A}}\int_{-\infty}^{\infty}k^2e^{-k^2/A}dk = \frac{\hbar^2}{\sqrt{\pi A}}\Gamma\left(\tfrac{3}{2}\right)A^{3/2} = \frac{\hbar^2}{\sqrt{\pi A}}\tfrac{1}{2}\sqrt{\pi}A^{3/2} = \tfrac{1}{2}\hbar^2 A\,.$$

In summary, $\Delta x = 1/\sqrt{2A}$, $\Delta p = \hbar\sqrt{A/2}$, and $(\Delta x)(\Delta p) = \frac{1}{2}\hbar$, so the inequality is just barely satisfied. For part (b), we have

$$\bar{x} = \alpha\int_{-\infty}^{\infty}xe^{-2\alpha|x|}dx = \alpha\int_0^{\infty}xe^{-2\alpha x}dx + \alpha\int_{-\infty}^{0}xe^{2\alpha x}dx = \alpha\int_0^{\infty}xe^{-2\alpha x}dx - \alpha\int_0^{\infty}xe^{-2\alpha x}dx = 0\,,$$

$$(\Delta x)^2 = \alpha\int_{-\infty}^{\infty}x^2e^{-2\alpha|x|}dx = 2\alpha\int_0^{\infty}x^2e^{-2\alpha x}dx = \frac{2\alpha}{(2\alpha)^3}\Gamma(3) = \frac{2}{4\alpha^2} = \frac{1}{2\alpha^2}\,,$$

$$\bar{p} = \frac{\hbar\alpha(2\alpha)^2}{2\pi}\int_{-\infty}^{\infty}\frac{kdk}{(k^2+\alpha^2)^2} = \frac{2\hbar\alpha^3}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\alpha\tan\theta\,\alpha\sec^2\theta\,d\theta}{(\alpha^2\tan^2\theta+\alpha^2)^2} = \frac{2\hbar\alpha}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin\theta\cos\theta\,d\theta = 0\,,$$

$$(\Delta p)^2 = \frac{\alpha(2\alpha)^2}{2\pi}\int_{-\infty}^{\infty}\frac{(\hbar k)^2\,dk}{(k^2+\alpha^2)^2} = \frac{2\hbar^2\alpha^3}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\alpha^2\tan^2\theta\,\alpha\sec^2\theta\,d\theta}{(\alpha^2\tan^2\theta+\alpha^2)^2} = \frac{2\hbar^2\alpha^2}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin^2\theta\,d\theta$$

$$= \frac{2\hbar^2\alpha^2}{\pi}\frac{\pi}{2} = \hbar^2\alpha^2\,.$$

In summary, $\Delta x = 1/\alpha\sqrt{2}$, $\Delta p = \hbar\alpha$, and $(\Delta x)(\Delta p) = \hbar/\sqrt{2}$, which also works.

6. [10] A particle of mass *m* lies in the harmonic oscillator potential, given by $V(x) = \frac{1}{2}m\omega^2 x^2$. Later we will solve this problem exactly, but for now, we only want an approximate solution.

(a) [4] Let the uncertainty in the position be $\Delta x = a$. What is the corresponding minimum uncertainty in the momentum Δp? Write an expression for the total energy (kinetic plus potential) as a function of *a*.

By the uncertainty principle, $(\Delta x)(\Delta p) \geq \frac{1}{2}\hbar$, so if $\Delta x = a$, then $\Delta p \geq \hbar/2a$. The formula for the energy is

$$E = \frac{p^2}{2m} + V(x) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

Now, the minimum energy classically would occur when $p = 0$ and $x = 0$, but this is impossible quantum mechanically because we cannot know them exactly. Assuming x and p actually take on values approximately equal to their uncertainties, the corresponding energy would be

$$E \approx \frac{\hbar^2}{8ma^2} + \frac{1}{2}m\omega^2 a^2$$

(b) [6] Find the minimum of the energy function you found in (a), and thereby estimate the minimum energy (called zero point energy) for a particle in a harmonic oscillator. Your answer should be very simple.

To find the minimum energy, we simply take the derivative of the function we just found and set it to zero

$$0 = \frac{dE}{da} = -\frac{\hbar^2}{4ma^3} + m\omega^2 a,$$
$$\hbar^2 = 4m^2\omega^3 a^4$$
$$a^2 = \frac{\hbar}{2m\omega}$$

You now simply plug this back into the energy formula to find

$$E = \frac{\hbar^2}{8m}\frac{2m\omega}{\hbar} + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega} = \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \frac{1}{2}\hbar\omega$$

This answer is, in fact, exactly correct, and the derivation can be shown to be exact as well, but this is a coincidence special to the harmonic oscillator.