Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 4

4.2 [10] The commutation relations of the angular momentum operator L_z with the momentum operator P were to be worked out in chapter 3, problem 3.

(a) [3] Using these commutation relations, derive two non-trivial uncertainty relationships.

The commutation relations were

$$[L_z, P_x] = i\hbar P_y, \quad [L_z, P_y] = -i\hbar P_x, \quad [L_z, P_z] = 0$$

According to the generalized uncertainty relation, we therefore have

$$\Delta L_z \, \Delta P_x \geq \tfrac{1}{2}\hbar \left|\left\langle P_y \right\rangle\right|, \quad \Delta L_z \, \Delta P_y \geq \tfrac{1}{2}\hbar \left|\left\langle P_x \right\rangle\right|, \quad \Delta L_z \, \Delta P_z \geq 0$$

Since the left side of each of these expressions is the product of two positive numbers, the last inequality doesn't give us any information, but the other two inequalities suffice.

(b) [3] Show that if you are in an eigenstate of any observable, the uncertainty in that observable is zero.

Let A be any observable, and $|a\rangle$ a normalized eigenstate with $A|a\rangle = a|a\rangle$. Then

$$(\Delta A)^2 = \left\langle A^2 \right\rangle - \left\langle A \right\rangle^2 = \langle a|A^2|a\rangle - \langle a|A|a\rangle^2 = a\langle a|A|a\rangle - \left(a\langle a|a\rangle\right)^2 = a^2\langle a|a\rangle - a^2 = 0$$

(c) [4] Show that if you are in an eigenstate of L_z, then you must have $\left\langle P_x \right\rangle = \left\langle P_y \right\rangle = 0$.

According to our inequalities we found above, if we are in an eigenstate of L_z, then we have $\Delta L_z = 0$, and therefore

$$0 \geq \tfrac{1}{2}\hbar \left|\left\langle P_y \right\rangle\right|, \quad 0 \geq \tfrac{1}{2}\hbar \left|\left\langle P_x \right\rangle\right|.$$

Since absolute values are never negative, it follows that $\left|\left\langle P_y \right\rangle\right| = \left|\left\langle P_x \right\rangle\right| = 0$, which implies

$$\left\langle P_x \right\rangle = \left\langle P_y \right\rangle = 0.$$

4.3 [10] We will eventually discover that particles have spin, which is described by three operators $\mathbf{S} = (S_x, S_y, S_z)$ with commutation relations

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y$$

A particle in a magnetic field of magnitude *B* pointing in the *z*-direction will have Hamiltonian $H = -\mu B S_z$ where μ and *B* are constants.

(a) [5] Derive formulas for the time derivative of all three components of $\langle \mathbf{S} \rangle$

We use the standard formula for the time evolution of any operator, namely

$$\begin{aligned}
\frac{d}{dt}\langle S_x \rangle &= \frac{i}{\hbar}\langle [H, S_x] \rangle = -\frac{i\mu B}{\hbar}\langle [S_z, S_x] \rangle = -\frac{i\mu B}{\hbar} i\hbar \langle S_y \rangle = \mu B \langle S_y \rangle, \\
\frac{d}{dt}\langle S_y \rangle &= \frac{i}{\hbar}\langle [H, S_y] \rangle = -\frac{i\mu B}{\hbar}\langle [S_z, S_y] \rangle = -\frac{i\mu B}{\hbar}(-i\hbar)\langle S_x \rangle = -\mu B \langle S_x \rangle, \\
\frac{d}{dt}\langle S_z \rangle &= \frac{i}{\hbar}\langle [H, S_z] \rangle = -\frac{i\mu B}{\hbar}\langle [S_z, S_z] \rangle = -\frac{i\mu B}{\hbar} 0 = 0.
\end{aligned}$$

(b) [5] At time *t* = 0, the expectation values of $\langle \mathbf{S} \rangle$ are given by

$$\langle S_x \rangle_{t=0} = a, \quad \langle S_y \rangle_{t=0} = 0, \quad \langle S_z \rangle_{t=0} = b$$

Determine the expectation value $\langle \mathbf{S} \rangle_t$ at later times.

Since the time derivative of $\langle S_z \rangle$ vanishes, this will just remain constant. The expectation value of the other two operators, however, are related by

$$d\langle S_x \rangle / dt = \mu B \langle S_y \rangle, \quad d\langle S_y \rangle / dt = -\mu B \langle S_x \rangle.$$

One way to proceed is to take another derivative of the first equation, which yields

$$d^2 \langle S_x \rangle / dt^2 = \mu B \, d\langle S_y \rangle / dt = -\mu^2 B^2 \langle S_x \rangle.$$

This suggests solutions along the lines of

$$\langle S_x \rangle = \alpha \cos(\mu B t) + \beta \sin(\mu B t).$$

Taking the derivative we can find

$$\langle S_y \rangle = -\alpha \sin(\mu B t) + \beta \cos(\mu B t).$$

Our boundary conditions at *t* = 0 then tell us that $\alpha = a$ and $\beta = 0$. In summary, we have

$$\langle S_x \rangle_t = a \cos(\mu B t), \quad \langle S_y \rangle_t = -a \sin(\mu B t), \quad \langle S_z \rangle_t = b.$$