Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 4

4.4 [15] A particle of mass *m* lies in a one-dimensional infinite square well in the region [0,*a*]. At *t* = 0, the wave function in the allowed region is

$$\langle x|\Psi(t=0)\rangle = \sqrt{30/a^5}\left(ax - x^2\right).$$

(a) [8] Write the wave function in the form $|\Psi(t=0)\rangle = \sum_n c_n |\phi_n\rangle$ **where** $|\phi_n\rangle$ **are the energy eigenstates. Check that the wave function is properly normalized, both in the original coordinate basis, and in the new basis, either analytically or numerically.**

The integral we need is

$$c_n = \langle \psi_n|\Psi(t=0)\rangle = \int_0^a \psi_n^*(x)\Psi(x,t=0)\,dx = \sqrt{\frac{2}{a}}\sqrt{\frac{30}{a^5}}\int_0^a \left(ax-x^2\right)\sin\left(\frac{\pi n x}{a}\right)dx$$

$$= \frac{2\sqrt{15}}{a^3}\left[-\frac{a}{\pi n}\left(ax-x^2\right)\cos\left(\frac{\pi n x}{a}\right) + \frac{a^2}{\pi^2 n^2}(a-2x)\sin\left(\frac{\pi n x}{a}\right) - \frac{2a^3}{\pi^3 n^3}\cos\left(\frac{\pi n x}{a}\right)\right]_0^a$$

$$= \frac{4\sqrt{15}}{\pi^3 n^3}\left[-\cos(\pi n)+1\right] = \frac{4\sqrt{15}}{\pi^3 n^3}\left[1-(-1)^n\right] = \begin{cases} 8\sqrt{15}/\pi^3 n^3 & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.}\end{cases}$$

So our initial state is

$$|\Psi(0)\rangle = \sum_n c_n|\phi_n\rangle = \sum_{n\text{ odd}} \frac{8\sqrt{15}}{\pi^3 n^3}|\phi_n\rangle.$$

In the original basis, normalization is given by

$$\langle\Psi(0)|\Psi(0)\rangle = \int_0^a \left|\langle x|\Psi(0)\rangle\right|^2 dx = \frac{30}{a^5}\int_0^a \left(ax-x^2\right)^2 dx = \frac{30}{a^5}\int_0^a \left(a^2x^2 - 2ax^3 + x^4\right)dx$$

$$= \frac{30}{a^5}\left(\tfrac{1}{3}a^2x^3 - \tfrac{1}{2}ax^4 + \tfrac{1}{5}x^5\right)\Big|_0^a = 1.$$

In the new basis, the normalization condition is

$$\langle\Psi(0)|\Psi(0)\rangle = \sum_n c_n^*\langle\phi_n|\sum_m c_m|\phi_m\rangle = \sum_n\sum_m c_n^* c_m\langle\phi_n|\phi_m\rangle = \sum_n |c_n|^2 = \frac{960}{\pi^6}\sum_{n\text{ odd}}\frac{1}{n^6} = \frac{960}{\pi^6}(1.0014)$$
$$= 1.0000.$$

I did the sum numerically, keeping only three terms, or you can do it analytically using Maple

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> sum(1/(2*n+1)^6,n=0..infinity);
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(b) [5] Find the expectation value of the Hamiltonain H for this wave function at $t = 0$, both in the original coordinate basis and the eigenstate basis. Check that they are the same, either analytically or numerically.

In the original basis, we simply use the Hamiltonian, but in the allowed region the potential vanishes, so

$$\langle H\rangle = \frac{1}{2m}\langle P^2\rangle = \frac{-\hbar^2}{2m}\frac{30}{a^5}\int_0^a \left(ax-x^2\right)\frac{d^2}{dx^2}\left(ax-x^2\right)dx = \frac{30\hbar^2}{ma^5}\int_0^a \left(ax-x^2\right)dx = \frac{30\hbar^2}{ma^5}\left(\tfrac{1}{2}ax^2-\tfrac{1}{3}x^3\right)\Big|_0^a$$
$$= \frac{5\hbar^2}{ma^2}.$$

In the final basis, we have

$$\langle H\rangle = \langle\Psi(0)|H|\Psi(0)\rangle = \sum_n c_n^*\langle\phi_n|H\sum_m c_m|\phi_m\rangle = \sum_n E_m c_n^*\sum_m c_m\langle\phi_n|\phi_m\rangle = \sum_n |c_n|^2 E_n$$
$$= \sum_{n\text{ odd}}\frac{960}{\pi^6 n^6}\frac{\pi^2\hbar^2 n^2}{2ma^2} = \frac{480\hbar^2}{\pi^4 ma^2}\sum_{n\text{ odd}}\frac{1}{n^4} = \frac{480\hbar^2}{\pi^4 ma^2}(1.0147) = \frac{4.9999\hbar^2}{ma^2}.$$

I did the final sum numerically, including ten terms. Obviously, it worked pretty well.

(c) [2] Write the wave function $|\Psi(t)\rangle$ at all times.

At arbitrary time, the wave function is

$$|\Psi(t)\rangle = \sum_n c_n|\phi_n\rangle\exp\left(-iE_n t/\hbar\right) = \sum_{n\text{ odd}}\frac{8\sqrt{15}}{\pi^3 n^3}|\phi_n\rangle\exp\left(-\frac{i\pi^2\hbar n^2}{2ma^2}t\right).$$