Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 6

6.2 [15] A particle of mass *m* in two dimensions is governed by the Hamiltonian

$$H=\frac{1}{2m}\left(P_x^2+P_y^2\right)+\tfrac{1}{4}\alpha\left(X^2+Y^2\right)^2+\tfrac{1}{3}\gamma\left(X^3-3XY^2\right)$$

(a) [5] Show that the Hamiltonian is invariant under the transformation $R\left(\mathcal{R}\left(\tfrac{2}{3}\pi\right)\right)$.

Under a rotation by $\tfrac{2}{3}\pi$, the coordinates transform as

$$\begin{pmatrix}X'\\Y'\end{pmatrix}=\begin{pmatrix}\cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3}\\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3}\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}X-\frac{\sqrt{3}}{2}Y\\ \frac{\sqrt{3}}{2}X-\frac{1}{2}Y\end{pmatrix}$$

We simply substitute this into our potential and check if it remains unchanged. We find

$$X'^2+Y'^2=\left(\tfrac{1}{2}X-\tfrac{\sqrt{3}}{2}Y\right)^2+\left(\tfrac{\sqrt{3}}{2}X+\tfrac{1}{2}Y\right)^2=\left(\tfrac{1}{4}X^2-\tfrac{\sqrt{3}}{2}XY+\tfrac{3}{4}Y^2\right)+\left(\tfrac{3}{4}X^2+\tfrac{\sqrt{3}}{2}XY+\tfrac{1}{4}Y^2\right)=X^2+Y^2$$

So this term is unchanged. The other term is more complicated, but with a little work, we see that

$$\begin{aligned}X'^3-3X'Y'^2&=\left(-\tfrac{1}{2}X-\tfrac{\sqrt{3}}{2}Y\right)^3-3\left(-\tfrac{1}{2}X-\tfrac{\sqrt{3}}{2}Y\right)\left(\tfrac{\sqrt{3}}{2}X-\tfrac{1}{2}Y\right)^2\\&=\left(-\tfrac{1}{8}X^3-\tfrac{3\sqrt{3}}{8}X^2Y-\tfrac{9}{8}XY^2-\tfrac{3\sqrt{3}}{8}Y^3\right)+\left(\tfrac{3}{2}X+\tfrac{3\sqrt{3}}{2}Y\right)\left(\tfrac{3}{4}X^2-\tfrac{\sqrt{3}}{2}XY+\tfrac{1}{4}Y^2\right)\\&=-\tfrac{1}{8}X^3-\tfrac{3\sqrt{3}}{8}X^2Y-\tfrac{9}{8}XY^2-\tfrac{3\sqrt{3}}{8}Y^3+\tfrac{9}{8}X^3+\tfrac{3\sqrt{3}}{8}X^2Y-\tfrac{15}{8}XY^2+\tfrac{3\sqrt{3}}{8}Y^3=X^3-3XY^2\,.\end{aligned}$$

Once again, this expression is unchanged, so we have proven our claim that this is unchanged under such a transformation. Hence this potential is invariant under rotations by 120°.

(b) [4] Classify the states according to their eigenvalues under $R\left(\mathcal{R}\left(\tfrac{2}{3}\pi\right)\right)$. What eigenvalues are possible?

Because the operator $R\left(\mathcal{R}\left(\tfrac{2}{3}\pi\right)\right)$ commutes with the Hamiltonian, our eigenstates of the Hamiltonian can be chosen to also be eigenstates of $R\left(\mathcal{R}\left(\tfrac{2}{3}\pi\right)\right)$. If we define $R\left(\mathcal{R}\left(\tfrac{2}{3}\pi\right)\right)|\psi\rangle=\lambda|\psi\rangle$, then as always since we have a unitary operator, λ must be a complex number of magnitude one. However, it is further restricted since three successive rotations are identical with no rotation, so we have

$$\lambda^3|\psi\rangle=\left[R\left(\mathcal{R}\left(\tfrac{2}{3}\pi\right)\right)\right]^3|\psi\rangle=R\left(\mathcal{R}\left(2\pi\right)\right)|\psi\rangle=R(1)|\psi\rangle=|\psi\rangle\,.$$

So we have $\lambda^3=1$. We can find the three roots in a variety of ways, the easiest being to factor it and use the quadratic equation:

$$\lambda^3 = 1 \quad \Rightarrow \quad 0 = \lambda^3 - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1) \quad \Rightarrow \quad \lambda = 1 \text{ or } \lambda = -\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}.$$

(c) [3] Suppose that $\psi(x,y)$ is an eigenstate of *H* and $R(\mathcal{R}(120°))$ with eigenvalues *E* and λ respectively. Show that $\psi^*(x,y)$ is also an eigenstate of *H* and $R(\mathcal{R}(120°))$, and determine its eigenvalues. (*E* is, of course, real).

Working in the coordinate representation, Schrödinger's equation and our symmetry relationship are

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$$
$$\psi\left(\mathcal{R}\left(\tfrac{2}{3}\pi\right)\mathbf{r}\right) = \lambda\psi(\mathbf{r})$$

Taking the complex conjugate of these relations, we see that

$$-\frac{\hbar^2}{2m}\nabla^2\psi^*(\mathbf{r}) + V(\mathbf{r})\psi^*(\mathbf{r}) = E\psi^*(\mathbf{r})$$
$$\psi^*\left(\mathcal{R}\left(\tfrac{2}{3}\pi\right)\mathbf{r}\right) = \lambda^*\psi^*(\mathbf{r})$$

In other words, the complex conjugate is also an eigenstate of *H* and *R* with eigenvalues *E* and λ* respectively.

(d) [3] Careful measurements of the Hamiltonian discovers that the system has some non-degenerate eigenstates (like the ground state), and some states that are two-fold degenerate (two eigenstates with exactly the same eigenvalue). Explain why these degeneracies are occurring.

Any state that has a complex value of λ must come with another state that has eigenvalue λ*. This will result in two-fold degeneracies. The non-degenerate states correspond to when λ = 1.