Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 7

3. [5] Show that the Pauli matrices, given by (7.17), satisfy

(a) [2] $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})^2 = 1$ **for any unit vector** $\hat{\mathbf{r}}$

Let $\hat{\mathbf{r}} = (x, y, z)$, with $x^2 + y^2 + z^2 = 1$. Then

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})^2 = (x\sigma_x + y\sigma_y + z\sigma_z)^2 = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} = \begin{pmatrix} z^2+x^2+y^2 & 0 \\ 0 & x^2+y^2+z^2 \end{pmatrix} = 1$$

(b) [3] $\exp\left(-\tfrac{1}{2}i\theta\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right) = \cos\left(\tfrac{1}{2}\theta\right) - i\sin\left(\tfrac{1}{2}\theta\right)(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})$

$$\begin{aligned}\exp\left(-\tfrac{1}{2}i\theta\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\tfrac{1}{2}i\theta\right)^n(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})^n = \sum_{n\text{ even}}\frac{1}{n!}\left(-\tfrac{1}{2}i\theta\right)^n + (\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\sum_{n\text{ odd}}\frac{1}{n!}\left(-\tfrac{1}{2}i\theta\right)^n \\ &= 1 - \tfrac{1}{2!}\left(\tfrac{1}{2}\theta\right)^2 + \tfrac{1}{4!}\left(\tfrac{1}{2}\theta\right)^4 - \tfrac{1}{6!}\left(\tfrac{1}{2}\theta\right)^6 + \cdots + i(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\left[-\left(\tfrac{1}{2}\theta\right) + \tfrac{1}{3!}\left(\tfrac{1}{2}\theta\right)^3 - \tfrac{1}{5!}\left(\tfrac{1}{2}\theta\right)^5 + \cdots\right] \\ &= \cos\left(\tfrac{1}{2}\theta\right) - i\sin\left(\tfrac{1}{2}\theta\right)(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\end{aligned}$$

4. **[10] Two particles have Hamiltonian $H_{\text{tot}} = \mathbf{P}_1^2/2m_1 + \mathbf{P}_2^2/2m_2 + V\left(\left|\mathbf{R}_1 - \mathbf{R}_2\right|\right)$, where the P's and R's have commutation relations**

$$\left[R_{ai}, P_{bj}\right] = i\hbar\delta_{ab}\delta_{ij}, \quad \left[R_{ai}, R_{bj}\right] = \left[P_{ai}, P_{bj}\right] = 0.$$

(a) [4] Show that if we define the four new vector operators $\left\{\mathbf{R}_{\text{cm}}, \mathbf{P}_{\text{cm}}, \mathbf{R}, \mathbf{P}\right\}$ as given in (7.38), they satisfy the commutation relations

$$\left[R_{\text{cm},i}, P_{\text{cm},j}\right] = \left[R_i, P_j\right] = i\hbar\delta_{ij}, \quad \left[R_{\text{cm},i}, P_j\right] = \left[R_i, P_{\text{cm},j}\right] = 0.$$

This is pretty straightforward: we simply write them down and work them out:

$$\left[R_{\text{cm},i}, P_{\text{cm},j}\right] = \frac{\left[m_1 R_{1i} + m_2 R_{2i}, P_{1j} + P_{2j}\right]}{m_1 + m_2} = \frac{m_1\left[R_{1i}, P_{1j}\right] + m_2\left[R_{2i}, P_{2j}\right]}{m_1 + m_2} = \frac{m_1 + m_2}{m_1 + m_2} i\hbar\delta_{ij} = i\hbar\delta_{ij},$$

$$\left[R_i, P_j\right] = \frac{\left[R_{1i} - R_{2i}, m_2 P_{1j} - m_1 P_{2j}\right]}{m_1 + m_2} = \frac{m_2\left[R_{1i}, P_{1j}\right] + m_2\left[R_{2i}, P_{2j}\right]}{m_1 + m_2} = \frac{m_2 + m_1}{m_1 + m_2} i\hbar\delta_{ij} = i\hbar\delta_{ij},$$

$$\left[R_i, P_{\text{cm},j}\right] = \left[R_{1i} - R_{2i}, P_{1j} + P_{2j}\right] = \left[R_{1i}, P_{1i}\right] - \left[R_{1i}, P_{2j}\right] = i\hbar\delta_{ij} - i\hbar\delta_{ij} = 0,$$

$$\left[R_{\text{cm},i}, P_j\right] = \frac{\left[m_1 R_{1i} + m_2 R_{2i}, m_2 P_{1j} - m_1 P_{2j}\right]}{\left(m_1 + m_2\right)^2} = \frac{m_1 m_2}{\left(m_1 + m_2\right)^2}\left(\left[R_{1i}, P_{1i}\right] - \left[R_{2i}, P_{2j}\right]\right) = 0.$$

(b) [6] Show that the Hamiltonian can then be written $H_{\text{tot}} = \mathbf{P}_{\text{cm}}^2/2M + \mathbf{P}^2/2\mu + V\left(\left|\mathbf{R}\right|\right)$, where *M* and *μ* are given by (7.39).

The potential term is trivial. For the kinetic term, we first note that

$$\left(m_1 + m_2\right)\mathbf{P} + m_1\mathbf{P}_{\text{cm}} = m_2\mathbf{P}_1 - m_1\mathbf{P}_2 + m_1\mathbf{P}_1 + m_1\mathbf{P}_2 = \left(m_1 + m_2\right)\mathbf{P}_1 \quad \Rightarrow \quad \mathbf{P}_1 = \frac{m_1}{M}\mathbf{P}_{\text{cm}} + \mathbf{P},$$

$$m_2\mathbf{P}_{\text{cm}} - \left(m_1 + m_2\right)\mathbf{P} = m_2\mathbf{P}_1 + m_2\mathbf{P}_2 - m_2\mathbf{P}_1 + m_1\mathbf{P}_2 = \left(m_1 + m_2\right)\mathbf{P}_2 \quad \Rightarrow \quad \mathbf{P}_2 = \frac{m_2}{M}\mathbf{P}_{\text{cm}} - \mathbf{P}.$$

We now substitute these in for each of the kinetic terms.

$$\begin{aligned} H &= \frac{1}{2m_1}\left(\frac{m_1}{M}\mathbf{P}_{\text{cm}} + \mathbf{P}\right)^2 + \frac{1}{2m_2}\left(\frac{m_2}{M}\mathbf{P}_{\text{cm}} - \mathbf{P}\right)^2 + V\left(\left|\mathbf{R}\right|\right) \\ &= \frac{m_1}{2M^2}\mathbf{P}_{\text{cm}}^2 + \frac{1}{M}\mathbf{P}_{\text{cm}}\cdot\mathbf{P} + \frac{1}{2m_1}\mathbf{P}^2 + \frac{m_2}{2M^2}\mathbf{P}_{\text{cm}}^2 - \frac{1}{M}\mathbf{P}_{\text{cm}}\cdot\mathbf{P} + \frac{1}{2m_2}\mathbf{P}^2 + V\left(\left|\mathbf{R}\right|\right) \\ &= \frac{m_1 + m_2}{2M^2}\mathbf{P}_{\text{cm}}^2 + \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{P}^2 + V\left(\left|\mathbf{R}\right|\right) = \frac{\mathbf{P}_{\text{cm}}^2}{2M} + \frac{\mathbf{P}^2}{2\mu} + V\left(\left|\mathbf{R}\right|\right). \end{aligned}$$

where in the last step we defined $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$, equivalent to the given definition.