Physics 741 – Graduate Quantum Mechanics 1

Final Exam, Fall 2015

Each question is worth 25 points, with points for each part marked separately. Some possibly useful formulas appear at the end of the test.

1. Charmonium is a bound state of a charm quark (spin $s_1 = \frac{1}{2}$) and anti-charm quark (spin $s_2 = \frac{1}{2}$). Although there are many bound states of charmonium, the lowest energy states have orbital angular momentum l = 0 or l = 1.
   (a) [3] What are the possible values for the total spin s of charmonium, corresponding to the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$?
   (b) [8] For each of the cases in part (a), and for each of the cases l = 0 or l = 1, what is the possible value of j of charmonium, corresponding to the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2$?
   (c) [10] Suppose one of the states found in part (b) is found to be an eigenstate of J_z with eigenvalue $+2\hbar$. Predict what possible eigenvalues the following operators might have: $\mathbf{J}^2, \mathbf{L}^2, L_z, \mathbf{S}^2, S_z, S_{1z}, S_{2z}$.
   (d) [4] Is the charm quark a fermion or boson? What about the anti-charm quark? What about the composite charmonium particles?

2. A sodium atom in the $3P_{3/2}$ state decays by emitting a photon to end up in the $2S_{1/2}$ state. The rate for the decay to a particular polarization can be written in terms of the matrix elements

$$\left\langle 3S, \tfrac{1}{2}, m \middle| R_q^{(1)} \middle| 3P, \tfrac{3}{2}, m' \right\rangle$$

   where $\frac{1}{2}$ and $\frac{3}{2}$ denote the total angular quantum number j for the initial and final atom, m and m' correspond to the z-component of the initial and final angular momenta, and $R_q^{(1)}$ are a set of three rank-one spherical tensors, with $q \in \{-1, 0, 1\}$.
   (a) [13] Give a complete list of the values of the triplets (m, q, m') such that this matrix element that does not vanish.
   (b) [12] Suppose the reduced matrix element were known, so that $\left\langle 3S, \tfrac{1}{2} \middle\| R^{(1)} \middle\| 3P, \tfrac{3}{2} \right\rangle = A$. Write the matrix elements from part (a) in terms of A and appropriate Clebsch-Gordan coefficients.

3. An electron is in spin state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\phi}|-\rangle\right)$, but the angle ϕ, is unknown, and uniformly distributed between 0 and π. Note that spin operators and Pauli matrices appear among the helpful formulas on the formula sheets.
   (a) [8] Compute the state operator (also known as density matrix) in matrix form. Check that the trace is correct.
   (b) [9] Calculate the expectation values of the three spin operators $\langle \mathbf{S} \rangle$.
   (c) [8] Show that if the Hamiltonian is of the form $H = \hbar\omega\sigma_y$, then ρ is time-independent.

4. In one of your problem sets, you showed that a spinless particle of mass m in a potential $V(\mathbf{r}) = \frac{1}{2} m\omega^2 \mathbf{r}^2$ has eigenstates $|n,l,m\rangle$ with energy $E = \hbar\omega\left(n + \frac{3}{2}\right)$, and l and m are the quantum numbers corresponding to L^2 and L_z respectively. In *this* problem, an electron (which has spin) of mass m will be influenced by scalar and vector potentials

$$U(\mathbf{R}) = -\tfrac{1}{2}\alpha\mathbf{R}^2 \quad \text{and} \quad \mathbf{A}(\mathbf{R}) = \tfrac{1}{2}\beta\left(X\hat{\mathbf{y}} - Y\hat{\mathbf{x}}\right)$$

   (a) [5] Find the electric and magnetic fields at all points.
   (b) [8] For an electron, write down the full Hamiltonian, expanding it out in powers of β. Show that some of the terms proportional to β can be written in terms of L_z.
   (c) [6] If $\beta = 0$, find the eigenstates and energies of the resulting Hamiltonian.
   (d) [6] Suppose $\beta \neq 0$, but assume β is small enough that the β^2 terms can be dropped. Show that the eigenstates of part (c) are still eigenstates. Find the new energy eigenvalues.

5. Two spinless non-interacting particles have mass m and are both in an infinite square well with allowed region $0 < x < a$. The one-particle eigenstate wave functions are $\psi_n(x) = \sqrt{2/a}\,\sin(n\pi x/a)$, which have energy $E_n = \pi^2 n^2 \hbar^2 / (2ma^2)$.
   (a) [6] If the two particles are non-identical particles, write explicitly the wave function $\psi(x_1, x_2)$ if the first one has n = 1 and the second has n = 2. What is the energy of this state?
   (b) [9] If the two particles are identical, write explicitly the wave function if one has n = 1 and the other has n = 2, if they are (i) bosons or (ii) fermions.
   (c) [10] For each of the three cases (non-identical, bosons, or fermions) find the probability density that both particles are simultaneously at $x = \frac{1}{3}a$. If the particles actually repel each other, so they don't like to be at the same place, which case would probably result in the lowest energy?

6. Consider N identical non-interacting spin-1/2 particles in a one-dimensional harmonic oscillator with potential $V(x) = \frac{1}{2} m\omega^2 x^2$.
   (a) [3] What are the energy eigenstates for a single particle? If we take spin into account, does this energy change, and how many particles fit into each state?
   (b) [12] If we have N particles in the ground state, which states will be occupied? Define the Fermi energy as the average of the highest occupied state and the first unoccupied state. What is the Fermi energy? Work this out in both the even and the odd case.
   (c) [10] What is the exact total energy for all the particles? For this part, assume N is even. Write your answer in the form $E_{tot} \propto NE_F$

Possibly Useful Sums: $\sum_{p=0}^{M-1} 1 = M$, $\sum_{p=0}^{M-1} p = \frac{1}{2}M(M-1)$, $\sum_{p=0}^{M-1} p^2 = \frac{1}{6}M(M-1)(2M-1)$,

Possibly Useful Formulas

Spin-½: $\mathbf{S} = \frac{1}{2}\hbar\boldsymbol{\sigma}$

$$, \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

EM Fields

$\mathbf{B} = \nabla\times\mathbf{A}$

$\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla U$

State Operator

$\rho = \sum_i f_i |\psi_i\rangle\langle\psi_i|$

$i\hbar\frac{d\rho}{dt} = [H,\rho]$

Wigner-Eckart Theorem

$$\langle \alpha, j, m | T_q^{(k)} | \alpha', j', m'\rangle = \frac{1}{\sqrt{2j+1}}\langle \alpha, j \| T^{(k)} \| \alpha', j'\rangle \langle j', k; m', q | j, m\rangle$$

EM Hamiltonian

$$H = \frac{\boldsymbol{\pi}^2}{2m} - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S}$$

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