

Physics 741 – Graduate Quantum Mechanics 1
Midterm Exam, Fall 2015

Please note that some possibly helpful formulas and integrals appear on the second page. Note also that there is one problem on the second page. The number of points for each question is marked at the start, with points for each part marked separately.

- [20 points]** Consider the wave function $\psi(x) = N(x+ia)^{-2}$. This state, once properly normalized, has expectation values $\langle P \rangle = \frac{3}{2}\hbar a^{-1}$ and $\langle P^2 \rangle = 3\hbar^2 a^{-2}$.
 - [7] What is the correct normalization N ?
 - [7] What is $\langle X \rangle$ and $\langle X^2 \rangle$ for this state?
 - [6] Find the uncertainties Δx and Δp and show that they satisfy the uncertainty relation.

- [20 points]** A particle of mass m lies in the infinite square well with allowed region $0 < x < a$. At $t = 0$, the wave function takes the form $\Psi(x, t = 0) = N \sin\left(\frac{2\pi x}{a}\right) \cos\left(\frac{5\pi x}{a}\right)$ in the allowed region and it vanishes elsewhere.
 - [7] Write this state in the form $|\Psi(0)\rangle = \sum_n c_n |\phi_n\rangle$. Some helpful integrals are provided.
 - [6] Determine the normalization constant N such that $|\Psi\rangle$ is normalized.
 - [7] Write $|\Psi(t)\rangle$ as a function of time in terms of the eigenstate basis, and write $\Psi(x, t)$.

- [25 points]** In a certain basis, the Hamiltonian takes the form $H = \hbar\omega \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$.
 - [12] Find the eigenvalues and normalized eigenvectors of this Hamiltonian.
 - [7] At $t = 0$, the state is in the state $|\Psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find $|\Psi(t)\rangle$ at all times.
 - [6] At time t , if we were to measure the energy, what would be the possible outcomes and corresponding probabilities?

- [15 points]** Consider the harmonic oscillator with mass m and angular frequency ω .
 - [7] For which non-negative integers q will the matrix elements $\langle 47 | X^q | 50 \rangle$ or $\langle 47 | P^q | 50 \rangle$ be non-zero? I want a complete rule that lets me tell when they are non-zero.
 - [8] For the smallest q for which they do not vanish, compute them.

5. [20 points] Bottomonium consists of a bottom quark and bottom anti-quark, each of mass m_b , bound by a potential that is approximately $V(r) = Ar$, where r is the distance between them.
- [4] Find a formula for the reduced mass of this system in terms of the quark mass m_b .
 - [5] What is an appropriate choice of coordinates for this system? Name two operators that commute with each other and with the Hamiltonian. Label the eigenstates of H by their eigenvalues under these two new operators. What can you tell me about these eigenvalues?
 - [5] Factor the wave function into an angular and a radial part. Describe completely one of these functions.
 - [6] For the remaining function, write an ordinary differential equation for the function. Do not attempt to solve it.

Possibly Helpful Formulas:

<p><u>Generators of Symmetries</u> $T(\mathbf{a}) = \exp(-i\mathbf{a} \cdot \mathbf{P}/\hbar)$ $R(\mathcal{R}(\hat{\mathbf{n}}, \theta)) = \exp(-i\theta \hat{\mathbf{n}} \cdot \mathbf{L}/\hbar)$</p>	<p><u>Harmonic Oscillator</u> $X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$ $P = i\sqrt{\frac{1}{2}\hbar m\omega}(a^\dagger - a)$ $a n\rangle = \sqrt{n} n-1\rangle$ $a^\dagger n\rangle = \sqrt{n+1} n+1\rangle$</p>	<p><u>\mathbf{P}^2 in Spherical or Cylindrical Coords.</u> $\mathbf{P}^2\psi = -\frac{\hbar^2}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} L_z^2 \psi + P_z^2 \psi$ $\mathbf{P}^2\psi = -\frac{\hbar^2}{r} \frac{\partial^2}{\partial r^2} (r\psi) + \frac{1}{r^2} \mathbf{L}^2 \psi$</p>
<p>Possibly Helpful Integrals: Definite Integrals: n, m and p are assumed to be positive integers</p> $\int_0^a \sin\left(\frac{\pi nx}{a}\right) \sin\left(\frac{\pi mx}{a}\right) dx = \int_0^a \cos\left(\frac{\pi nx}{a}\right) \cos\left(\frac{\pi mx}{a}\right) dx = \frac{1}{2} a \delta_{nm}$ $\int_0^a \sin\left(\frac{\pi nx}{a}\right) \sin\left(\frac{\pi mx}{a}\right) \cos\left(\frac{\pi px}{a}\right) dx = \frac{1}{4} a (\delta_{n,m+p} + \delta_{m,n+p} - \delta_{p,n+m})$ $\int_0^a \cos\left(\frac{\pi nx}{a}\right) \cos\left(\frac{\pi mx}{a}\right) \cos\left(\frac{\pi px}{a}\right) dx = \frac{1}{4} a (\delta_{n,m+p} + \delta_{m,n+p} + \delta_{p,n+m})$ $\int_{-\infty}^{\infty} \frac{x^n dx}{(x^2 + b^2)^m} = 0 \quad \text{if } n \text{ is odd,} \quad \int_{-\infty}^{\infty} \frac{x^{2n} dx}{(x^2 + b^2)^m} = \infty \quad \text{if } n \geq m, \quad \int_{-\infty}^{\infty} \frac{dx}{x^2 + b^2} = \frac{\pi}{b},$ $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)^2} = \frac{\pi}{2b^3}, \quad \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + b^2)^2} = \frac{\pi}{2b}, \quad \int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)^4} = \frac{5\pi}{16b^7}, \quad \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + b^2)^4} = \frac{\pi}{16b^5}.$ <p>Indefinite Integral: $\int (x + ib)^n dx = \frac{1}{n+1} (x + ib)^{n+1} + C$</p>		<p><u>Infinite Square Well</u> $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi nx}{a}\right)$ $E_n = \frac{\pi^2 \hbar^2 n^2}{2ma^2}$</p>
		<p><u>Red. Mass</u> $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$</p>