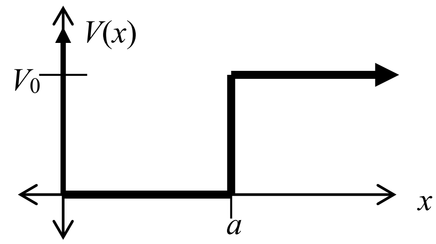
Physics 741 – Graduate Quantum Mechanics 1

Midterm Exam, Fall 2022

Please note that some possibly helpful formulas and integrals appear on the second page. Note also that there is one problem on the second page. Each question is worth 20 points, with points for each part marked separately.

1. A particle of mass m lies in a potential $V(x)$, where

$$V(x)=\begin{cases}\infty & x<0\\ 0 & 0<x<a\\ V_0 & x>a\end{cases}$$



This potential is sketched at right. We will attempt to find bound states, $0 < E < V_0$.

(a) [7] For the region $0 < x < a$, write the most general solution of Schrödinger's time independent equation, and relate any parameters to the energy E, applying appropriate boundary conditions at $x = 0$.

(b) [7] Repeat for the region $x > a$, applying appropriate boundary conditions at $x = \infty$.

(c) [6] What boundary conditions can you at $x = a$? Take the ratio (or substitute) these equations to find a relation between the parameters found in parts (a) and (b), cancelling any normalization factors. You do not need to solve these equations.

2. A quantum mechanical system is two dimensional, and in a choice of basis, the Hamiltonian is $H=\hbar\omega\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}$ and $B=b\begin{pmatrix}4 & 2\\ 2 & 1\end{pmatrix}$. At $t = 0$, it is in the state $\left|\Psi(t=0)\right\rangle=\begin{pmatrix}1\\ 0\end{pmatrix}$.

(a) [7] For the operator B, find the eigenvalues and normalized eigenvectors.

(b) [4] At time $t = 0$, the system is measured using the operator B. What is the probability that the result comes out the more positive eigenvalue? Assuming it does, what is the state immediately after the measurement?

(c) [5] After the measurement described in part (b), what is the state vector at general time t? What is it at the specific time $\omega t = \pi$?

(d) [4] At this time, B is measured again. What is the probability *this time* that the result will be the more positive eigenvalue?

3. A particle of mass m lies in the potential $V(x,y,z)=\alpha\left(x^4+y^4+z^4\right)+\beta\left(x^2z-y^2z\right)+\gamma xyz$.

Consider the rotation operator that rotates by 90° around the z-axis and then flips z, so that $\mathcal{R}(x,y,z)=(y,-x,-z)$, *i.e.* $x'=y$, $y'=-x$, $z'=-z$.

(a) [6] Show that this is a symmetry operation; that is, V is unchanged by this transformation.

(b) [6] Argue that if this symmetry operation were performed a particular number of times, the resulting symmetry operation would correspond with the identity operation. How many times?

(c) [8] Argue that eigenstates of the Hamiltonian can be chosen to also be eigenstates of this symmetry operation. What are the possible eigenvalues of these states under the symmetry operation?

4. A particle is in the ground state of the infinite square well at time $t = 0$.
   (a) [10] Calculate the expectation values $\langle X \rangle$, $\langle X^2 \rangle$, $\langle P \rangle$ and $\langle P^2 \rangle$ for this state.
   (b) [5] Find the uncertainties Δx and Δp, and check the uncertainty relation.
   (c) [5] Suppose that *before* we did the measurement, we allowed the system to evolve under the influence of the Hamiltonian until an arbitrary time *t*. How would this change the answers to parts (a) and (b)?

5. A particle of mass *m* lies in the harmonic oscillator potential $V = \frac{1}{2} m\omega^2 x^2$. At $t = 0$, the wave function is given by $\Psi(x, t=0) = Nx^2 e^{-\alpha x^2/2}$, where $\alpha = m\omega/\hbar$.
   (a) [4] What is the normalization constant *N*? Some possibly helpful integrals are given below.
   (b) [6] Write this wave function in the form $|\Psi(t=0)\rangle = \sum_n c_n |\phi_n\rangle$, where $|\phi_n\rangle$'s are the eigenstates of the Hamiltonian. The explicit forms for the first three are given below. Check that the state vector is normalized in the new basis of the $|\phi_n\rangle$'s.
   (c) [5] Write the state vector $|\Psi(t)\rangle$ at all times.
   (d) [5] Find the probability density that the particle is at the origin $x = 0$ at all times. Simplify as much as possible.

Infinite Square well: mass *m*, region $0 < x < a$: $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right)$, $E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$.

Harmonic Oscillator Wave Functions:

$$\phi_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}, \quad \phi_1(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} \sqrt{2\alpha}\, x e^{-\alpha x^2/2}, \quad \phi_2(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{2}} \left(2\alpha x^2 - 1\right) e^{-\alpha x^2/2}.$$

Possibly Helpful Integrals: *n* is assumed to be a positive integer, and *A* is positive

$$\int_{-\infty}^{\infty} x^n e^{-Ax^2} dx = \begin{cases} \Gamma\left(\frac{n+1}{2}\right) A^{-(n+1)/2} & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}, \quad \Gamma\left(\tfrac{7}{2}\right) = \tfrac{15}{8}\sqrt{\pi}, \quad \Gamma\left(\tfrac{9}{2}\right) = \tfrac{105}{16}\sqrt{\pi}, \quad \Gamma\left(\tfrac{11}{2}\right) = \tfrac{945}{32}\sqrt{\pi}.$$

$$\int_0^a \sin^2\left(\frac{\pi n x}{a}\right) dx = \int_0^a \cos^2\left(\frac{\pi n x}{a}\right) dx = \frac{a}{2}, \quad \int_0^a \sin\left(\frac{\pi n x}{a}\right) \cos\left(\frac{\pi n x}{a}\right) dx = 0,$$

$$\int_0^a x \sin^2\left(\frac{\pi n x}{a}\right) dx = \int_0^a x \cos^2\left(\frac{\pi n x}{a}\right) dx = \frac{a^2}{4}, \quad \int_0^a x \sin\left(\frac{\pi n x}{a}\right) \cos\left(\frac{\pi n x}{a}\right) dx = -\frac{a^2}{4\pi n},$$

$$\int_0^a x^2 \sin^2\left(\frac{\pi n x}{a}\right) dx = \left(\frac{1}{6} - \frac{1}{4\pi^2 n^2}\right) a^3, \quad \int_0^a x^2 \cos^2\left(\frac{\pi n x}{a}\right) dx = \left(\frac{1}{6} + \frac{1}{4\pi^2 n^2}\right) a^3.$$