Physics 742 – Graduate Quantum Mechanics 2

Solutions to Second Exam, Spring 2024

Each question is worth 20 points. Some possibly useful formulas appear below or on the handout

1. A particle of mass μ and wave number k scatters from a potential given by $V(r) = \begin{cases} \alpha/r & r < R, \\ 0 & r > R, \end{cases}$ where α is small. Find the differential cross-section using the first Born approximation. I recommend doing the $\cos\theta$ integral before doing the radial integral.

We will need to perform the Fourier transform that appears in the formulas. We note that since the potential is spherically symmetric, it depends only on the *magnitude* of **K**, so we can work this out pretending that **K** is in the *z*-direction, and then $\mathbf{K}\cdot\mathbf{r} = Kr\cos\theta$. We have

$$\int d^3\mathbf{r}\, V(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} = \int_0^\infty V(r) r^2 dr \int_0^{2\pi} d\phi \int_{-1}^{1} e^{-iKr\cos\theta} d(\cos\theta) = 2\pi \int_0^R \frac{\alpha}{r} r^2 dr \frac{1}{-iKr} e^{-iKr\cos\theta}\Big|_{\cos\theta=-1}^{\cos\theta=1}$$
$$= \frac{2\pi}{-iK}\int_0^R \alpha\left(e^{-iKr} - e^{iKr}\right) dr = \frac{2\pi\alpha}{(-iK)^2} e^{-iKr}\Big|_0^R - \frac{2\pi\alpha}{(-iK)(iK)} e^{iKr}\Big|_0^R$$
$$= -\frac{2\pi\alpha}{K^2}\left(e^{-iKR} - 1 + e^{iKR} - 1\right) = \frac{4\pi\alpha}{K^2}\left[1 - \cos(KR)\right].$$

Substituting this into the formula for the first Born approximation, we have

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4}\left|\int d^3\mathbf{r}\, V(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2 = \frac{\mu^2}{4\pi^2\hbar^4}\left[\frac{4\pi\alpha}{K^2}\left(1 - \cos(KR)\right)\right]^2 = \frac{4\mu^2\alpha^2}{\hbar^4 K^4}\left[1 - \cos(KR)\right]^2$$

We can then use the expression $\mathbf{K}^2 = 2k^2(1 - \cos\theta)$ to rewrite this as

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2\alpha^2}{\hbar^4 k^4 (1 - \cos\theta)^2}\left(1 - \cos\left(kR\sqrt{2 - 2\cos\theta}\right)\right)^2$$

An obscure formula from trigonometry, $1 - \cos\theta = 2\sin^2\left(\tfrac{1}{2}\theta\right)$, can make this a little better looking:

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2\alpha^2 \sin^4\left(kR\sin\left(\tfrac{1}{2}\theta\right)\right)}{\hbar^4 k^4 \sin^4\left(\tfrac{1}{2}\theta\right)}.$$

2. A particle of mass *m* in 1D in a delta-function potential, $V(x) = -(\hbar^2\lambda/m)\delta(x)$ has a single bound state whose wave function is $\psi(x) = \sqrt{\lambda}e^{-\lambda|x|}$. Find the probability that it will remain in the bound state at all times if (a) the value of λ is increased to 4λ in a single sudden step, or (b) the value of λ increases in *two* sudden steps, first to 2λ and then to 4λ. In each case write your final answer as a numerical probability. What do you think would happen if it increased in an infinite number of tiny steps from λ to 4λ?

As λ increases, the wave function always takes the same *form*, but with the value of λ gradually changing, so for example the bound state when $\lambda \to 4\lambda$ will become $\psi(x) = \sqrt{4\lambda}e^{-4\lambda|x|}$. The probability will be just

$$P(|\psi_\lambda\rangle \to |\psi_{4\lambda}\rangle) = |\langle\psi_{4\lambda}|\psi_\lambda\rangle|^2 = \left|\sqrt{4\lambda}\sqrt{\lambda}\int_{-\infty}^{\infty} e^{-\lambda x}e^{-4\lambda x}dx\right|^2 = \left|2\cdot 2\lambda\int_0^{\infty} e^{-5\lambda x}dx\right|^2 = \left|\frac{4\lambda\cdot 1!}{5\lambda}\right|^2$$

$$= \frac{16}{25} = 64.00\%.$$

Now, for a *single* step of going from $\lambda \to 2\lambda$, this probability is given by

$$P(|\psi_\lambda\rangle \to |\psi_{2\lambda}\rangle) = |\langle\psi_{2\lambda}|\psi_\lambda\rangle|^2 = \left|\sqrt{2\lambda}\sqrt{\lambda}\int_{-\infty}^{\infty} e^{-\lambda x}e^{-2\lambda x}dx\right|^2 = \left|2\sqrt{2}\lambda\int_0^{\infty} e^{-3\lambda x}dx\right|^2 = \left|\frac{2\sqrt{2}\lambda\cdot 1!}{3\lambda}\right|^2 = \frac{8}{9}.$$

However, this is for just one step. Noting that λ is canceling out, it is easy to see that

$$P(|\psi_{2\lambda}\rangle \to |\psi_{4\lambda}\rangle) = P(|\psi_\lambda\rangle \to |\psi_{2\lambda}\rangle) = \frac{8}{9}$$

as well. So the probability of the two steps both occurring will be

$$P(|\psi_\lambda\rangle \to |\psi_{2\lambda}\rangle \to |\psi_{4\lambda}\rangle) = P(|\psi_\lambda\rangle \to |\psi_{2\lambda}\rangle)\cdot P(|\psi_{2\lambda}\rangle \to |\psi_{4\lambda}\rangle) = \frac{8}{9}\cdot\frac{8}{9} = \frac{64}{81} = 79.01\%.$$

The probability has increased, even though we had to make two transitions, and demanded that it remain in the bound state at each step.

Now it is not obvious how this continues as we go to higher order, but it is at least suggestive. If you do it in four steps, you will find that the probability climbs to 88.73%; in eight, to 97.05%. More importantly, if you do it in a very large number of tiny steps, then in fact you are doing the change adiabatically, and in the adiabatic approximation, the probability is 100%, so it makes sense that this series of steps converges to this limiting value.

3. A particle of mass *m* is in the ground state $|0\rangle$ of the 1D harmonic oscillator with frequency ω. It is then subjected to a series of 25 perturbative pulses at intervals *T* of the form $W = \sum_{p=0}^{24} AX\delta(t - pT)$. Find a general formula to leading order in *A* that it transitions to some other state $|n\rangle$ as a function of *T* (for which a sum may be left undone), and evaluate it specifically (do the sum) in the cases $\omega T = \pi$ and $\omega T = 2\pi$.

We use the first-order transition matrix, which we compute only for a final state $|n\rangle \neq |0\rangle$, and is given by

$$S_{n0} = \delta_{n0} + \frac{1}{i\hbar}\int dt\, W_{n0}(t) e^{i\omega_{n0}t} = 0 + \frac{1}{i\hbar}\int_0^\infty dt \langle n| AX \sum_{p=0}^{24} \delta(t - pT)|0\rangle e^{i\omega nt}$$

$$= \frac{A}{i\hbar}\sqrt{\frac{\hbar}{2m\omega}} \langle n|(a + a^\dagger)|0\rangle \sum_{p=0}^{24} e^{i\omega npT} = \frac{A}{i\sqrt{2m\omega\hbar}} \langle n|1\rangle \sum_{p=0}^{24} e^{i\omega npT} = \frac{A}{i\sqrt{2m\omega\hbar}} \delta_{n1} \sum_{p=0}^{24} e^{i\omega pT}.$$

We used that fact that $|n\rangle \neq |0\rangle$ to get rid of the first term, used the delta-function to do the time integrals, and at the last step, used the fact that only $n = 1$ contributes to finish the final sum. We then simply use the fact that the probability is the square of this amplitude to conclude that

$$P(|0\rangle \to |1\rangle) = |S_{n0}|^2 = \frac{A^2}{2m\omega\hbar}\left|\sum_{p=0}^{24} e^{i\omega pT}\right|^2.$$

This sum is a geometric series, which can be summed in general, but we weren't asked to do this.

In the two specific cases, the exponentials are easy to simplify, and the resulting sum is easy to do, so we have

$$\omega T = \pi: \quad P(|0\rangle \to |1\rangle) = \frac{A^2}{2m\omega\hbar}\left|\sum_{p=0}^{24}\left(e^{i\omega T}\right)^p\right|^2 = \frac{A^2}{2m\omega\hbar}\left|\sum_{p=0}^{24}(-1)^p\right|^2 = \frac{A^2}{2m\omega\hbar}|1 - 1 + 1 - 1 + - \cdots + 1|^2$$

$$= \frac{A^2}{2m\omega\hbar},$$

$$\omega T = 2\pi: \quad P(|0\rangle \to |1\rangle) = \frac{A^2}{2m\omega\hbar}\left|\sum_{p=0}^{24}\left(e^{i\omega T}\right)^p\right|^2 = \frac{A^2}{2m\omega\hbar}\left|\sum_{p=0}^{24}1^p\right|^2 = \frac{A^2}{2m\omega\hbar}|1 + 1 + 1 + 1 + \cdots + 1|^2$$

$$= \frac{25^2 A^2}{2m\omega\hbar} = \frac{625A^2}{2m\omega\hbar}.$$

We see that matching the period to the frequency results in a much larger probability; this illustrates the concept of resonance.

4. A system of pure photons is in a superposition of an arbitrary number of photon states, $|\Psi\rangle = A\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n |n,\mathbf{q},\tau\rangle$, where $c|\mathbf{q}| = cq = \omega$. Find the normalization factor A and the expectation value of the energy $\langle\Psi|H|\Psi\rangle$. Some useful sums appear on the equation sheet.

We want the state to be normalized, so we have

$$1 = \langle\Psi|\Psi\rangle = |A|^2 \sum_{m=0}^{\infty}\left(\tfrac{4}{5}\right)^m \langle m,\mathbf{q},\tau|\sum_{n=0}^{\infty}\left(\tfrac{4}{5}\right)^n |n,\mathbf{q},\tau\rangle = |A|^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\tfrac{4}{5}\right)^{m+n} \langle m,\mathbf{q},\tau|n,\mathbf{q},\tau\rangle$$

$$= |A|^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\tfrac{4}{5}\right)^{m+n} \delta_{mn} = |A|^2 \sum_{n=0}^{\infty}\left(\tfrac{16}{25}\right)^n = |A|^2 \frac{1}{1-\frac{16}{25}} = \frac{25}{9}|A|^2 .$$

Up to an arbitrary phase, the normalization is $A = \frac{3}{5}$.

To find the expectation value of the energy, we simply add in the Hamiltonian. We know that each of these states are eigenstates of the Hamiltonian with energy $E = n\hbar\omega$, so we have

$$\langle\Psi|H|\Psi\rangle = |A|^2 \sum_{m=0}^{\infty}\left(\tfrac{4}{5}\right)^m \langle m,\mathbf{q},\tau|H\sum_{n=0}^{\infty}\left(\tfrac{4}{5}\right)^n |n,\mathbf{q},\tau\rangle = \left(\tfrac{3}{5}\right)^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\tfrac{4}{5}\right)^{m+n} n\hbar\omega\langle m,\mathbf{q},\tau|n,\mathbf{q},\tau\rangle$$

$$= \frac{9}{25}\hbar\omega\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} n\left(\tfrac{4}{5}\right)^{m+n}\delta_{mn} = \frac{9}{25}\hbar\omega\sum_{n=0}^{\infty} n\left(\tfrac{16}{25}\right)^n = \frac{9}{25}\hbar\omega\frac{\frac{16}{25}}{\left(1-\frac{16}{25}\right)^2} = \frac{9}{25}\hbar\omega\frac{16\cdot 25}{\left(25-16\right)^2}$$

$$= \frac{9\cdot 16}{9^2}\hbar\omega = \frac{16}{9}\hbar\omega.$$

5. An electron of mass *m* is in the $|1,1,0\rangle$ state of the 3D *asymmetric* harmonic oscillator with potential $V(x,y,z)=\frac{1}{2}m\omega_0^2\left(x^2+4y^2+9z^2\right)$. What is the energy of the state $|n,p,q\rangle$? Find the rate of decay Γ_{FI} in the dipole approximation for each possible final state.

We note that the frequency in the three directions are unequal. Matching to the standard form $\frac{1}{2}m\omega^2x^2$ in each of the three directions, we see that $\omega_x=\omega_0$, $\omega_y=2\omega_0$, and $\omega_z=3\omega_0$. The energy of the state $|n,p,q\rangle$ is therefore

$$\varepsilon_{npq}=\hbar\omega_0\left(n+\tfrac{1}{2}\right)+2\hbar\omega_0\left(p+\tfrac{1}{2}\right)+3\hbar\omega_0\left(q+\tfrac{1}{2}\right)=\hbar\omega_0\left(n+2p+3q+3\right).$$

We wish to go to a lower energy state, and the three position operators *X*, *Y*, and *Z* can only raise or lower one of the three labels by one. Since we must go down in energy, and we can only change one of them by one, the final state must be $|1,0,0\rangle$ or $|0,1,0\rangle$. When calculating the dipole moments, we must remember that the frequency has to be modified accordingly. We have

$$\mathbf{r}_{100,110}=\langle 1,0,0|\mathbf{R}|1,1,0\rangle=\langle 1,0,0|Y\hat{\mathbf{y}}|1,1,0\rangle=\hat{\mathbf{y}}\sqrt{\frac{\hbar}{2m(2\omega_0)}}\langle 1,0,0|\left(a_y+a_y^\dagger\right)|1,1,0\rangle=\hat{\mathbf{y}}\sqrt{\frac{\hbar}{4m\omega_0}},$$

$$\mathbf{r}_{010,110}=\langle 0,1,0|\mathbf{R}|1,1,0\rangle=\langle 0,1,0|X\hat{\mathbf{x}}|1,1,0\rangle=\hat{\mathbf{x}}\sqrt{\frac{\hbar}{2m\omega_0}}\langle 0,1,0|\left(a_y+a_y^\dagger\right)|1,1,0\rangle=\hat{\mathbf{x}}\sqrt{\frac{\hbar}{2m\omega_0}}.$$

For each of these states, we must also calculate the resulting frequency difference, which is

$$\omega_{110,100}=\frac{1}{\hbar}\left(\varepsilon_{110}-\varepsilon_{100}\right)=\frac{1}{\hbar}\left[\hbar\omega_0\left(1+2\cdot 1+3\cdot 0+3\right)-\hbar\omega_0\left(1+2\cdot 0+3\cdot 0+3\right)\right]=2\omega_0,$$

$$\omega_{110,010}=\frac{1}{\hbar}\left(\varepsilon_{110}-\varepsilon_{010}\right)=\frac{1}{\hbar}\left[\hbar\omega_0\left(1+2\cdot 1+3\cdot 0+3\right)-\hbar\omega_0\left(0+2\cdot 1+3\cdot 0+3\right)\right]=\omega_0.$$

We then substitute these into the rate for spontaneous decay, which yields

$$\Gamma(110\to 100)=\frac{4\alpha}{3c^2}\omega_{110,100}^3\left|\mathbf{r}_{110,100}\right|^2=\frac{4\alpha}{3c^2}(2\omega_0)^3\left(\sqrt{\frac{\hbar}{4m\omega_0}}\right)^2=\frac{4\alpha 8\omega_0^3\hbar}{3c^2 4m\omega_0}=\frac{8\alpha\hbar\omega_0^2}{3mc^2},$$

$$\Gamma(110\to 010)=\frac{4\alpha}{3c^2}\omega_{110,010}^3\left|\mathbf{r}_{110,010}\right|^2=\frac{4\alpha}{3c^2}(\omega_0)^3\left(\sqrt{\frac{\hbar}{2m\omega_0}}\right)^2=\frac{4\alpha\omega_0^3\hbar}{3c^2 2m\omega_0}=\frac{2\alpha\hbar\omega_0^2}{3mc^2}.$$

We were not asked to calculate the branching ratio, but it is easy to see that it will be $\frac{4}{5}$ and $\frac{1}{5}$ for the two states respectively.

Possibly Helpful Formulas:

Spontaneous Decay	1st Born Approximation	1D harmonic oscillator:
$\Gamma = \frac{4\alpha}{3c^2}\omega_{IF}^3 \left\|\mathbf{r}_{FI}\right\|^2$	$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4}\left\|\int d^3\mathbf{r}\, V(\mathbf{r})\, e^{-i\mathbf{K}\cdot\mathbf{r}}\right\|^2$ $\mathbf{K}^2 = 2k^2(1-\cos\theta)$	$V(x) = \frac{1}{2}m\omega^2 x^2$ $X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$ $a\|n\rangle = \sqrt{n}\|n-1\rangle$ $a^\dagger\|n\rangle = \sqrt{n+1}\|n+1\rangle$
	Time-dependent Perturbation Theory $S_{FI} = \delta_{FI} + \frac{1}{i\hbar}\int dt\, W_{FI}(t)\, e^{i\omega_{FI}t} + \cdots$	

Trigonometry: $e^{i\theta} + e^{-i\theta} = 2\cos\theta, \quad e^{i\theta} - e^{-i\theta} = 2i\sin\theta.$

Integrals: $\int x^n e^{-Ax} dx = -\left(\frac{1}{A} + \frac{nx}{A^2} + \frac{n(n-1)}{A^3}x^2 + \cdots + \frac{n!}{A^{n+1}}\right)e^{-Ax}, \quad \int_0^\infty x^n e^{-Ax} dx = \frac{n!}{A^{n+1}}$

Sums: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}, \quad \sum_{n=0}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}, \quad \text{if } |x| < 1.$