PHY 742 Spring 2025 First Exam Name ___________________

The value of each question is listed in square brackets at the start of the problem or part.

1. [15] A spin-½ particle has state operator $\rho = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. You will also need $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
   (a) [6] What are the eigenvalues of this state operator? What restrictions on *a* and *b* are necessary for it to be a valid state operator? For what value(s) is it a pure state?
   (b) [4] If we measured $S_x = \frac{1}{2}\hbar\sigma_x$, what would be the expectation value?
   (c) [5] If the Hamiltonian is $H = \omega S_x$, for what values would ρ be time-independent?

2. [15] This problem should be worked entirely in terms of the Heisenberg formulation. A particle is described exclusively in terms of its spin. The Hamiltonian is given by $H = \omega S_z$.
   (a) [5] Find in this formalism equations for all three time derivatives $\frac{d}{dt} S_i(t)$.
   (b) [2] Argue that one of the three is time independent, *i.e.* $S_C(t) = S_C(0)$.
   (c) [8] Check that the other two differential equations can be satisfied by the equations $S_A(t) = \cos(\omega t) S_A(0) + \sin(\omega t) S_B(0)$ and $S_B(t) = \cos(\omega t) S_B(0) - \sin(\omega t) S_A(0)$.

3. [15] A particle of mass *m* in one dimension is in the potential $V(x) = \begin{cases} \frac{1}{2} m\omega_1^2 x^2 & x < 0, \\ \frac{1}{2} m\omega_2^2 x^2 & x > 0. \end{cases}$
   Using the WKB method, estimate the energy of the *n*'th eigenstate.

4. [15] A particle of mass *m* in one dimension is in the potential $V(x) = B|x|$. Estimate the ground state energy by the variational method using the trial wave function $\psi(x) = e^{-Ax^2/2}$.

5. [25] A particle of mass *m* lies in the 2D infinite square square well with allowed region $0 < x < a$ and $0 < y < a$. In addition, there is a perturbation $W(x, y) = \gamma\delta\left(x - \frac{1}{4}a\right)\delta\left(y - \frac{1}{4}a\right)$.
   (a) [3] What are the exact eigenstates and energies in the limit of no perturbation, $\gamma = 0$?
   (b) [7] Find the ground state energy to first order in γ.
   (c) [15] For the first excited states, find the eigenstates to leading (zeroth) order and energies to first order.

6. [15] Suppose we model a hydrogen atom as an electron orbiting a nucleus that produces a Coulomb potential given by $V(r) = -\frac{k_e e^2}{r}\left(1 - e^{-r/R}\right)$, where *R* is the nuclear size.
   (a) [4] If we compare this to the standard assumption that the nucleus has zero size, what is the perturbation *W* introduced by using a finite size?
   (b) [11] Assume the unperturbed wave function $\psi_{nlm}(\mathbf{r})$ is slowly varying over the nuclear scale. Estimate the energy shift ε' due to finite nuclear size in terms of $\psi_{nlm}(\mathbf{r})$. For which values of *n*, *l*, and *m* will the result be non-zero?

Possibly Helpful Formulas:

Heisenberg Picture	State Operators	1D Infinite Square Well	Spin Commutators
$\frac{d}{dt}A(t)=\frac{i}{\hbar}\left[H(t),A(t)\right]$	$\rho=\sum_i f_i\vert\psi_i\rangle\langle\psi_i\vert$ $i\hbar\frac{d}{dt}\rho=\left[H,\rho\right]$	$\psi_n(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{\pi nx}{a}\right)$ $E_n=\frac{\pi^2\hbar^2n^2}{2ma^2}$	$\left[S_x,S_y\right]=i\hbar S_z$ $\left[S_y,S_z\right]=i\hbar S_x$ $\left[S_z,S_x\right]=i\hbar S_y$

Possibly Helpful Integrals: In the equations below A and B are positive.

$$\int_0^\infty x^n e^{-Ax}dx=\frac{\Gamma(n+1)}{A^{n+1}},\quad \int_0^\infty x^n e^{-Ax^2}dx=\frac{\Gamma\left(\frac{n+1}{2}\right)}{2A^{\frac{n+1}{2}}},$$

$$\Gamma(n)=(n-1)!\text{ if } n \text{ is an integer},$$

$$\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi},\quad \Gamma\left(\tfrac{3}{2}\right)=\tfrac{1}{2}\sqrt{\pi},\quad \Gamma\left(\tfrac{5}{2}\right)=\tfrac{3}{4}\sqrt{\pi},$$

$$\int\sqrt{A-B^2x^2}\,dx=\frac{A}{2B}\sin^{-1}\left(\frac{Bx}{\sqrt{A}}\right)+\frac{x}{2}\sqrt{A-B^2x^2}$$

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