Physics 742 – Graduate Quantum Mechanics 2

Last Exam, Spring 2021

Please note that some possibly helpful formulas are listed on the next page. Each question is worth twenty points. The points for individual parts are marked in []'s.

1. A particle with mass μ and momentum $\hbar k$ moving in the z-direction scatters from a weak potential $V(x,y,z) = V_0 e^{-x^2/\alpha} e^{-y^2/\alpha} \delta(z)$. Find the differential and total cross-section using the first Born approximation. For the total cross section, you may leave one integral unfinished.

2. A particle of mass m lies in a potential $V(x) = \begin{cases} \frac{1}{2} m\omega^2 x^2 & x > 0, \\ \infty & x < 0. \end{cases}$ It is in the ground state, whose wave function is given in the allowed region ($x > 0$) by $\psi_g(x) = \sqrt{2}\phi_1(x)$, where $\phi_1(x)$ is the first excited state of the *standard* harmonic oscillator (with $V(x) = \frac{1}{2} m\omega^2 x^2$). The barrier forcing $x > 0$ is then moved towards minus infinity, so that the potential becomes the standard harmonic oscillator. Find the probability of it ending up in the ground state or first excited state if the barrier is moved (a) adiabatically, or (b) suddenly.

3. A particle of mass m is in the ground state $|0\rangle$ at $t \to -\infty$ of a harmonic oscillator with potential $V(x) = \frac{1}{2} m\omega^2 x^2$, but there is also a perturbation of the form $W(x,t) = Bx^2 e^{-At^2}$, where B is small. To first order in perturbation theory, what *other* final states $|n\rangle$ can it end up in, and what is the corresponding probability in the limit $t \to +\infty$?

4. A system lies in a superposition of 1 or 2 photons, $|\psi\rangle = N\left(|1,\mathbf{q},\tau\rangle + i\sqrt{2}|2,\mathbf{q},\tau\rangle\right)$, where $\mathbf{q} = q\hat{\mathbf{x}}$ and $\boldsymbol{\varepsilon}_\tau = \hat{\mathbf{y}}$.
   (a) [3] What is the correct normalization constant N?
   (b) [17] What is the expectation value of the electric field for this state?

5. An electron of mass m is in a 3D symmetric harmonic oscillator with frequency ω in the superposition state $|2,0,0\rangle$. Assume the dipole approximation is valid.
   (a) [5] To which state(s) will it decay? Calculate the relevant matrix elements $\mathbf{r}_{FI}$.
   (b) [7] Calculate the differential decay rates $d\Gamma_1/d\Omega$ and $d\Gamma_2/d\Omega$ to the two polarization states $\boldsymbol{\varepsilon}_1 = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$ and $\boldsymbol{\varepsilon}_2 = (-\sin\phi, \cos\phi, 0)$.
   (c) [8] Integrate over angles and determine the polarized decay rates Γ_1 and Γ_2, and the branching ratio to each of these polarizations.

1D Harmonic Oscillator: $X=\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^{\dagger}\right)$,

$$\phi_0\left(x\right)=\left(\alpha/\pi\right)^{1/4}e^{-\alpha x^2/2}\,,\quad \phi_1\left(x\right)=\left(\alpha/\pi\right)^{1/4}\sqrt{2\alpha}\,xe^{-\alpha x^2/2}\,,\quad \alpha=m\omega/\hbar.$$

First Order Time-dependent Perturbation Theory: $S_{FI}=\delta_{FI}+\frac{1}{i\hbar}\int^{T}W_{FI}\left(t\right)e^{i\omega_{FI}t}dt+\cdots$

Electric Field: $\mathbf{E}\left(\mathbf{r}\right)=\sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}i\left(a_{\mathbf{k}\sigma}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}e^{i\mathbf{k}\cdot\mathbf{r}}-a_{\mathbf{k}\sigma}^{\dagger}\boldsymbol{\varepsilon}_{\mathbf{k}\sigma}^{*}e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$

Spontaneous Decay: $\frac{d\Gamma}{d\Omega}=\frac{\alpha\omega_{IF}^3}{2\pi c^2}\left|\boldsymbol{\varepsilon}^{*}\cdot\mathbf{r}_{FI}\right|^2,\quad \Gamma=\frac{4\alpha\omega_{IF}^3}{3c^2}\left|\mathbf{r}_{FI}\right|^2$

Possibly Helpful Integrals: $\int_{-\infty}^{\infty}e^{-Ax^2-Bx}dx=\sqrt{\pi/A}\,e^{B^2/4A}$.

$$\int_0^{\infty}x^n e^{-Ax^2}dx=\tfrac{1}{2}A^{-(n+1)/2}\Gamma\left(\tfrac{n+1}{2}\right),\quad \Gamma\left(1\right)=\Gamma\left(2\right)=1,\quad \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}\,,\quad \Gamma\left(\tfrac{3}{2}\right)=\tfrac{1}{2}\sqrt{\pi}\,,\quad \Gamma\left(\tfrac{5}{2}\right)=\tfrac{3}{4}\sqrt{\pi}$$

$$\int_0^{2\pi}\sin^2\phi\,d\phi=\int_0^{2\pi}\cos^2\phi\,d\phi=\pi,\quad \int_0^{\pi}\sin^2\theta\,d\theta=\frac{\pi}{2},\quad \int_0^{\pi}\sin^3\theta\,d\theta=\frac{4}{3},\quad \int_0^{\pi}\cos^2\theta\sin\theta\,d\theta=\frac{2}{3}.$$