Physics 742 – Graduate Quantum Mechanics 2

Second Exam, Spring 2024

Each question is worth 20 points. Some possibly useful formulas appear below or on the handout

1. A particle of mass μ with wave number k scatters from a potential given by $V(r)=\begin{cases}\alpha/r & r<R,\\ 0 & r>R,\end{cases}$ where α is small. Find the differential cross-section using the first Born approximation. I recommend doing the $\cos\theta$ integral before doing the radial integral.

2. A particle of mass m in 1D in a delta-function potential, $V(x)=-(\hbar^2\lambda/m)\delta(x)$ has a single bound state whose wave function is $\psi(x)=\sqrt{\lambda}e^{-\lambda|x|}$. Find the probability that it will remain in the bound state at all times if (a) the value of λ is increased to 4λ in a single sudden step, or (b) the value of λ increases in *two* sudden steps, first to 2λ and then to 4λ. In each case write your final answer as a numerical probability. What do you think would happen if it increased in an infinite number of tiny steps from λ to 4λ?

3. A particle of mass m is in the ground state $|0\rangle$ of the 1D harmonic oscillator with frequency ω. It is then subjected to a series of 25 perturbative pulses at intervals T of the form $W=\sum_{p=0}^{24} AX\delta(t-pT)$. Find a general formula to leading order in A that it transitions to some other state $|n\rangle$ as a function of T (for which a sum may be left undone), and evaluate it specifically (do the sum) in the cases $\omega T=\pi$ and $\omega T=2\pi$.

4. A system of pure photons is in a superposition of an arbitrary number of photon states, $|\Psi\rangle=A\sum_{n=0}^{\infty}\left(\tfrac{4}{5}\right)^n|n,\mathbf{q},\tau\rangle$, where $c|\mathbf{q}|=cq=\omega$. Find the normalization factor A and the expectation value of the energy $\langle\Psi|H|\Psi\rangle$. Some useful sums appear on the equation sheet.

5. An electron of mass m is in the $|1,1,0\rangle$ state of the 3D *asymmetric* harmonic oscillator with potential $V(x,y,z)=\tfrac{1}{2}m\omega_0^2\left(x^2+4y^2+9z^2\right)$. What is the energy of the state $|n,p,q\rangle$? Find the rate of decay Γ_{FI} in the dipole approximation for each possible final state.

Possibly Helpful Formulas:

Spontaneous Decay

$$\Gamma=\frac{4\alpha}{3c^2}\omega_{IF}^3\left|\mathbf{r}_{FI}\right|^2$$

1st Born Approximation

$$\frac{d\sigma}{d\Omega}=\frac{\mu^2}{4\pi^2\hbar^4}\left|\int d^3\mathbf{r}\,V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2$$

$$\mathbf{K}^2=2k^2(1-\cos\theta)$$

Time-dependent Perturbation Theory

$$S_{FI}=\delta_{FI}+\frac{1}{i\hbar}\int_0^T dt\,W_{FI}(t)e^{i\omega_{FI}t}+\cdots$$

1D harmonic oscillator:

$$V(x)=\tfrac{1}{2}m\omega^2x^2$$

$$X=\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)$$

$$a|n\rangle=\sqrt{n}|n-1\rangle$$

$$a^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$$

Trigonometry: $e^{i\theta} + e^{-i\theta} = 2\cos\theta, \quad e^{i\theta} - e^{-i\theta} = 2i\sin\theta.$

Integrals: $\int x^n e^{-Ax} dx = -\left(\frac{1}{A} + \frac{nx}{A^2} + \frac{n(n-1)}{A^3}x^2 + \cdots + \frac{n!}{A^{n+1}}\right)e^{-Ax}, \quad \int_0^\infty x^n e^{-Ax} dx = \frac{n!}{A^{n+1}}$

Sums: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}, \quad \sum_{n=0}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}, \quad \text{if } |x| < 1.$

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