Lecture 5: Limit Laws and Continuity

Laws:

Suppose c is a constant and

$\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$

exist. Then,

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$

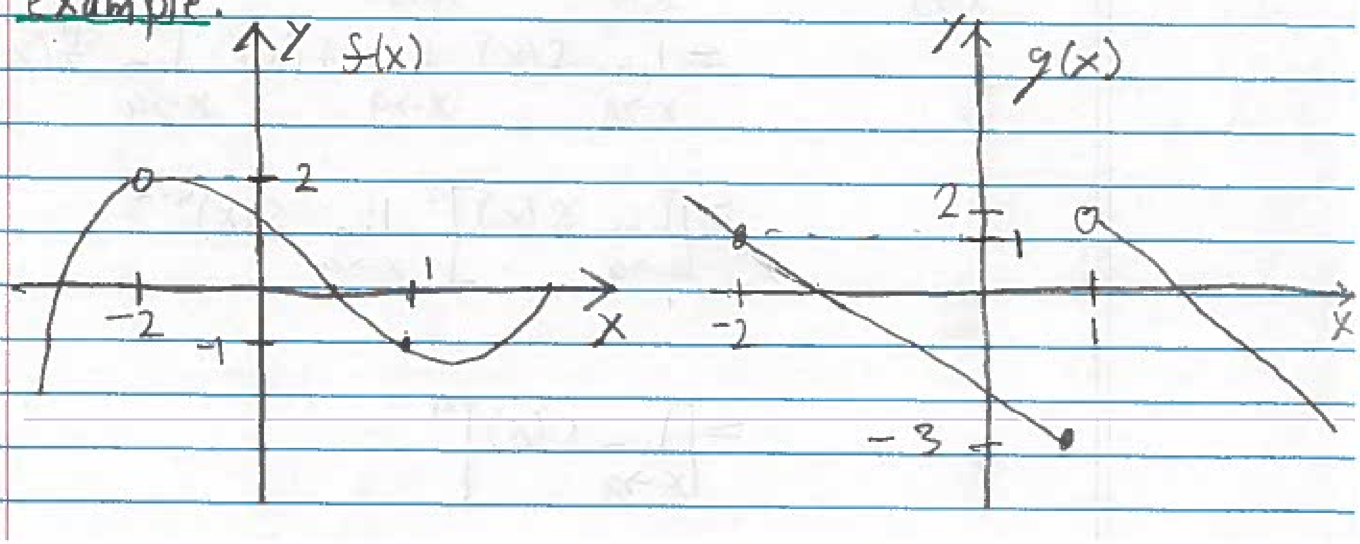
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$

3. $\lim_{x\to a}[c\, f(x)] = c \lim_{x\to a} f(x).$

4. $\lim_{x\to a}[f(x)\cdot g(x)] = f(a)\cdot g(a).$

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0.$

example:



1. $\lim_{x\to -2} [f(x)+5g(x)] = \lim_{x\to -2} f(x) + \lim_{x\to -2} 5g(x)$

$= \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x)$

$= 2 + 5\cdot 1$

$= 7$

2. $\lim_{x\to 1} f(x)\cdot g(x) = ?$

$\lim_{x\to 1^+} f(x)\cdot g(x) = \lim_{x\to 1^+} f(x)\cdot \lim_{x\to 1^+} g(x) = -1\cdot 2 = -2$

$\lim_{x\to 1^-} f(x)\cdot g(x) = \lim_{x\to 1^-} f(x)\cdot \lim_{x\to 1^+} g(x) = -1\cdot(-3) = 3$

$\Rightarrow$ limit does not exist.

More Laws:

1. $\lim_{x\to a} f(x)^n = \left[\lim_{x\to a} f(x)\right]^n$, n is an integer.

proof:

$\lim_{x\to a} [f(x)^n] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} f(x)^{n-1}$

$= \lim_{x\to a} f(x)\cdot \lim_{x\to a} f(x)\cdot \lim_{x\to a} f(x)^{n-2}$

$= \left[\lim_{x\to a} f(x)\right]^2 \lim_{x\to a} f(x)^{n-2}$

$\vdots$

$= \left[\lim_{x\to a} f(x)\right]^n.$

2. $\lim_{x \to a} c = c$

3. $\lim_{x \to a} x = a$

4. $\lim_{x \to a} x^n = a^n$.

~~proof~~:

$$\lim_{x \to a} x^n = \left[\lim_{x \to a} x\right]^n = a^n.$$

5. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

Theorem - $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

example:

1. $\lim_{x \to 0} |x| = \lim_{x \to 0^+} x = \lim_{x \to 0^-} (-x) = 0$.

2. $\lim_{x \to 0} \frac{x}{|x|}$?

$$\lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{x} = 1.$$

$$\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

The limit does not exist.

3. Find k so the following limit exists

$\lim_{x\to 4} f(x)$

if

$$f(x)=\begin{cases}\sqrt{x-4} & \text{if } x>4\\ k-2x & \text{if } x<4\end{cases}$$

$\lim_{x\to 4^+} f(x) = 0$

$\lim_{x\to 4^-} f(x) = k-8$

Therefore if $k=8$ it follows that

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x) = 0.$

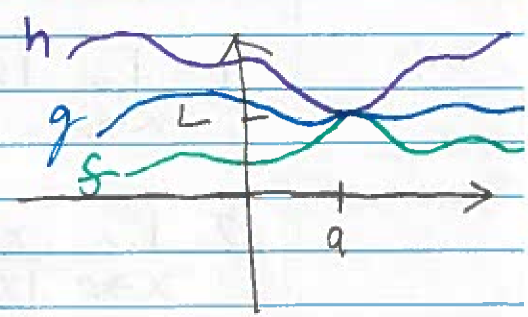
$\Rightarrow \lim_{x\to 4} f(x) = 0.$

Theorem - If $f(x)\le g(x)\le h(x)$ and

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

then

$\lim_{x\to a} g(x) = L.$

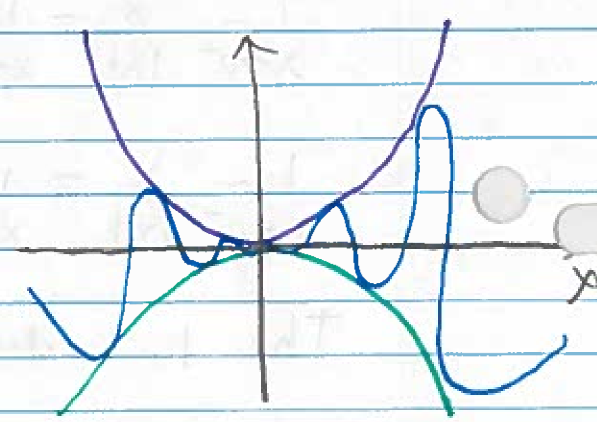


example:

1. $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$?

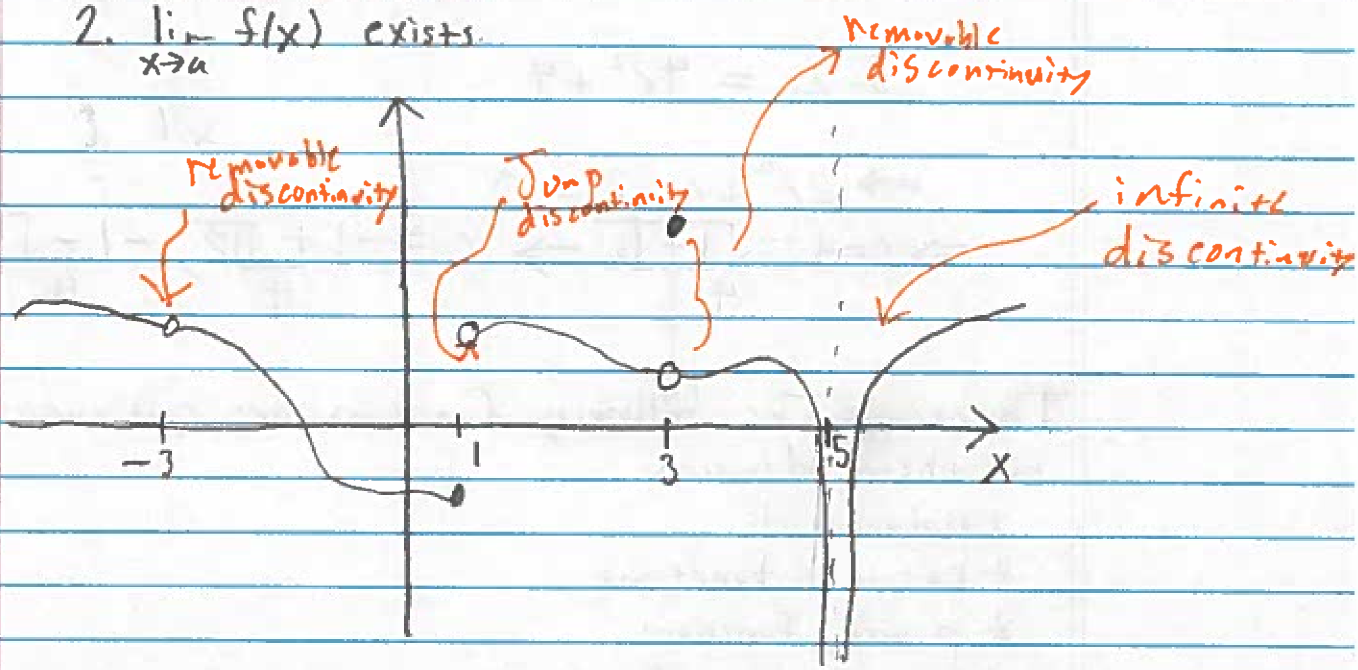
$-x^2 \le x^2\sin\left(\frac{1}{x}\right) \le x^2$

$\rightarrow \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$



Definition - A function f is continuous at a if
$$\lim_{x\to a} f(x) = f(a).$$

* This definition assumes
1. $f(a)$ is defined
2. $\lim_{x\to a} f(x)$ exists.



$x=-3$: Discontinuity, function is not defined
$x=1$: Limit does not exist, discontinuity
$x=3$: Limit does not match function value, discontinuity
$x=5$: Function is not defined

Definition - A function f is continuous on an interval if it is continuous at every number in the interval.

example:
Find a value of k which makes the following function continuous:
$$f(x) = \begin{cases} c^2x^2 + 2x, & \text{if } x<2 \\ x^3 - cx, & \text{if } x \geq 2 \end{cases}$$

$$\lim_{x \to 2^+} f(x) = 8 - 2c$$

$$\lim_{x \to 2^-} f(x) = 4c^2 + 4$$

If this function is continuous then:

$$8 - 2c = 4c^2 + 4$$

$$\Rightarrow 2c^2 + c - 2 = 0$$

$$\Rightarrow c = \frac{-1 \pm \sqrt{1 + 16}}{4} \Rightarrow c = \frac{-1 + \sqrt{17}}{4}, \frac{-1 - \sqrt{17}}{4}$$

Theorem - The following functions are continuous on their domains

* polynomials
* rational functions
* power functions
* trigonometric functions

Theorem - If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(b)$$

i.e,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Theorem – If $\lim_{x\to a} f(x) = b$ then $\lim_{x\to a} [f(x)]^n = b^n$.

proof:
Let $g(x) = x^n$. Then g is continuous and

$$\lim_{x\to a} [f(x)]^n = \lim_{x\to a} g(f(x))$$
$$= g\left(\lim_{x\to a} f(x)\right)$$
$$= g(b)$$
$$= b^n.$$

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Theorem – If g is continuous at a and f is continuous at $g(a)$, then the composite function $h(x) = f(g(x))$ is continuous at a.

proof:

$$\lim_{x\to a} h(x) = \lim_{x\to a} f(g(x))$$
$$= f\left(\lim_{x\to a} g(x)\right)$$
$$= f(g(a))$$
$$= h(a).$$

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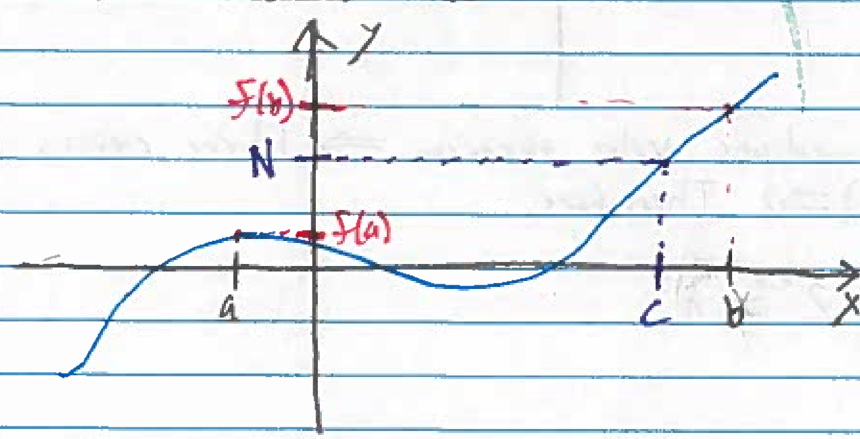
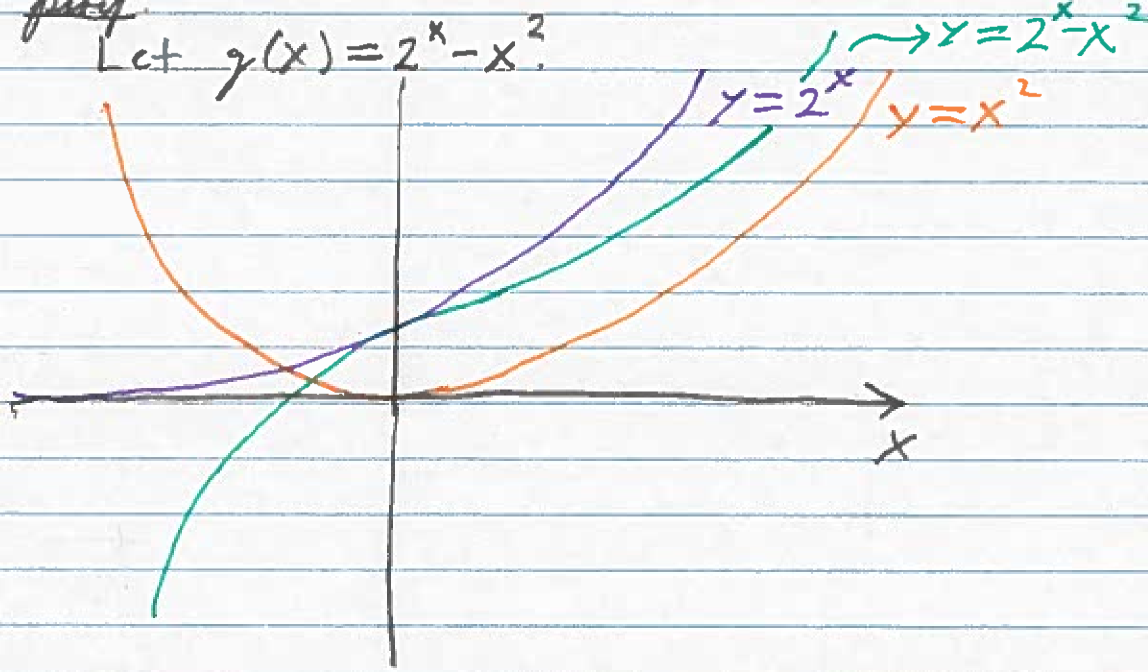
Theorem (The Intermediate Value Theorem)

Suppose that f is continuous on the interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exists c in $[a,b]$ such that $f(c) = N$.



example:
Show that there is a solution to the following equation $2^x = x^2$.

~~proof~~:
Let $g(x) = 2^x - x^2$.



$g(1) = 2 - 1 = 1$
$g(-1) = \frac{1}{2} - 1 = -\frac{1}{2}$

By the intermediate value theorem there exists c in $[-1, 1]$ such that $g(c) = 0$.