Math 113
Final Exam Problems

1. The vector field
$$\mathbf{F}(x,y,z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$$
is conservative. Find a potential function.

2. Let S be the union of two surfaces S_1 and S_2 where S_1 is the part of the sphere $x^2+y^2+z^2=1$ that satisfies $z \geq 0$ and S_2 is the disk $x^2+y^2 \leq 1$ with $z=0$. Evaluate the flux of the vector field
$$\mathbf{F}(x,y,z) = \langle xy^2, yz^2, zx^2 + \frac{1}{2}z^2 \rangle$$
across S.

3. Evaluate the following line integral:
$$\int_C (1+xy)\, e^{xy}\, dx + \left(e^y + x^2 e^{xy}\right)\, dy,$$
where the curve C is parametrized by $r(t) = \sin(t)\mathbf{i} + (1+t)\mathbf{j}$, $0 \leq t \leq \pi$.

4. Evaluate the work done by the vector field $\mathbf{F}(x,y) = \langle \sqrt{1+x^4}, xy \rangle$ moving a particle from $(0,0)$ to $(1,0)$ then to $(2,2)$, and then back to $(0,0)$, all along straight lines.

5. Consider the function
$$f(x,y) = \frac{1}{3}x^3 - 4xy + \frac{2}{3}y^3 + y.$$
Find the point (x_0, y_0) at which the rate of change of the function in the $\mathbf{i}+\mathbf{j}$ direction is the largest.

6. Evaluate the iterated integral:
$$\int_0^1 \int_x^1 e^{y^2}\, dy dx.$$

7. Find the volume below the surface $z = \sqrt{4+x^2+y^2}$ and above the disc $0 \leq x^2+y^2 \leq 1$, $z=0$.

8. Consider the planes $x+y+z=1$ and $x-y+z=2$. If θ is the angle between the planes, find $\cos(\theta)$. Find the equation of the plane that is orthogonal to the two given planes and passes through the point $(2,1,3)$.

9. Consider $f(x,y) = x^2 + \dfrac{y^2}{4}$. Find the directions of maximum increase and decrease at the point $(1,2)$. At the point $(1,2)$, find the directions in which the function is neither increasing or decreasing.

10. Let $f(x,y) = F(x^2+y^2) + G(xy)$ where F, G are functions of a single variable satisfying $F(2) = 1$, $F'(2) = 2$, $G(-1) = -1$, $G'(-1) = -2$. Calculate f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ at $(x,y) = (1,-1)$.

11. Consider the plane $x + 2y + 2z = 4$. Use Lagrange multipliers to find the distance between the plane and the origin.

12. The pressure P, volume V, and temperature T of one mole of an ideal gas satisfy $PV = RT$, where $R > 0$ is a constant. Suppose R is measured using $R = PV/T$.
    - Find the differential dR.
    - If the percentage errors in the measurement of P, V , and T are 1%, 2%, and 3%, respectively, find the maximum percentage error in R.

13. Consider the surface $xyz = 1$ and the point $P = (2,1,1/2)$. Find the normal to the surface at P. Find the equation of the tangent line at P.

14. Let C be the part of the circle $x^2+y^2 = 1$ from the point $(1,0)$ to the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Let

$$\mathbf{F}(x,y) = \langle y\cos(xy) + y, x\cos(xy) + x\rangle$$

Evaluate the integral

$$\int_C \mathbf{F}(x,y) \cdot dr.$$

15. Suppose that $f(x,y)$ is a function in two variables with continuous derivatives such that $f(1,2) = 5$. Suppose that it is known that $\nabla f(1,2) = (3,4)$. Find an equation of the tangent line to the level curve $f(x,y) = 5$ at $(x,y) = (1,2)$.

16. Let S be the part of the sphere $x^2+y^2+z^2 = 4$, $y \geq 0$. Evaluate the flux of the vector field

$$F(x,y,z) = (x - 2yz)\mathbf{i} + (y + xz)\mathbf{j} + (z + xy)\mathbf{k}$$

across S.

17. Evaluate the following integral:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} \sqrt{x^2+y^2}\, dzdydx.$$

18. The half cone $z = \sqrt{x^2+y^2}$ divides the ball $x^2+y^2+z^2 \leq 1$ into two parts. Evaluate the volume of the larger part.

19. Find the minimum of the function $f(x,y,z) = x^2+y^2+z^2$ assuming that the points (x,y,z) are on the surface $x+y+z = 12$.

20. Evaluate the integral

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dydx.$$

21. Evaluate the area of the part of the sphere $x^2+y^2+z^2 = 4$ that lies above the plane $z = 1$.

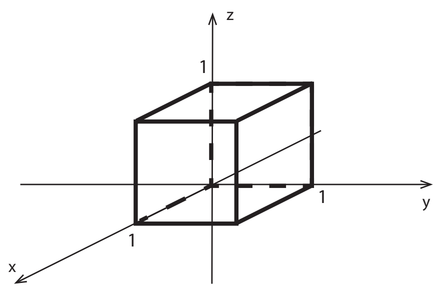
22. Evaluate the volume of the solid bounded by the surface $y = x^2$ and the planes $y + z = 2$ and $y - z = 0$.

23. Let $f(x, y)$ be a differentiable function. Suppose that the rate of change of f at the point $P = (1, 2)$ in the direction from P to the point $Q = (0, 1)$ is $\sqrt{2}$. Suppose furthermore that the directional derivative $D_{\mathbf{u}} f(1, 2) = 1$ where $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$. Find the partial derivatives f_x and f_y at P.

24. Consider the function $f(x, y, z) = x^2 + \frac{y^2}{2} + 2z^2 + 2xz$. Find all points on the level surface $f(x, y, z) = 4$ at which the tangent plane is parallel to the xy-plane.

25. Find the length of the curve $r(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle$, $0 \le t \le 1$.

26. Let
$$\mathbf{F}(x, y, z) = \langle x^2, 2xy + \sin(z), x \rangle.$$
Evaluate the flux of the vector field $\mathbf{F}$ across the boundary surface of the unit cube pictured below.

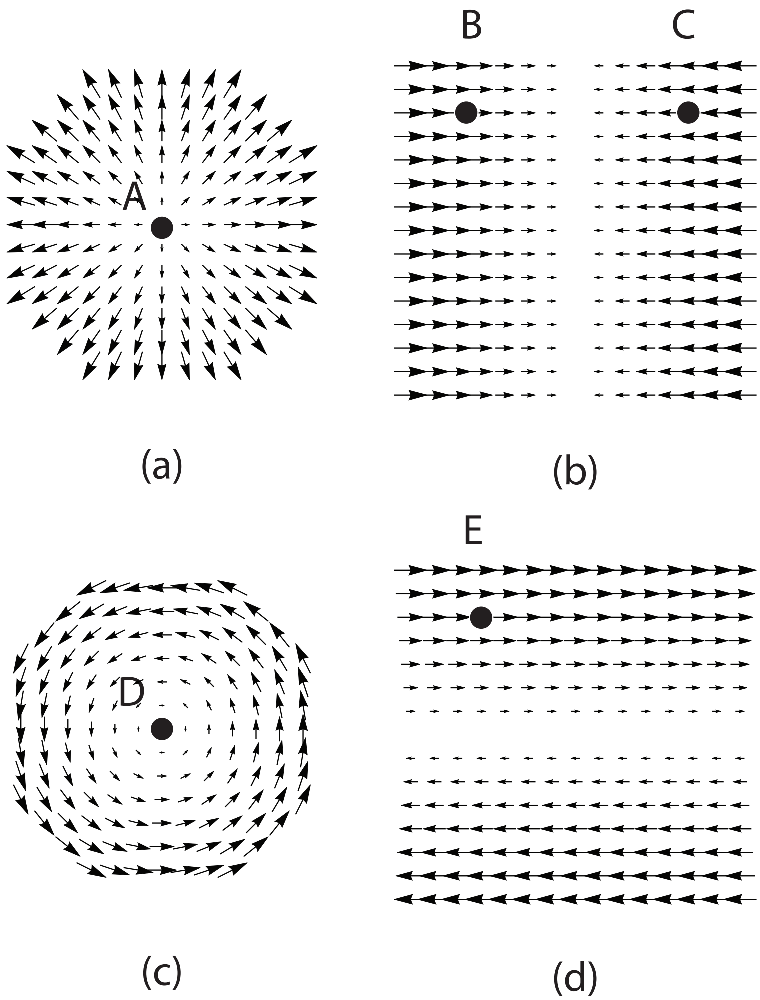


27. Let C be the part of the curve $x + y^2 = 4$ from the point $(-5, -3)$ to point $(0, 2)$. Evaluate the integral
$$\int_C y^2\, dx - 2xy\, dy.$$

28. Consider the vector field
$$\mathbf{F}(x, y, z) = \langle z^2 + y\sin(yz), 2xze^{z^2} - y - z, x^2 + y^2 + z \rangle.$$
It is known that $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$. Now consider the sphere $x^2 + y^2 + z^2 = 1$. The plane $z = -\frac{1}{2}$ divides the sphere into two parts. Let S be the smaller part that is below the plane. Evaluate the following integral:
$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

29. Evaluate the integral
$$\iint_S \langle x, y, 1 \rangle \cdot d\mathbf{S}$$
where S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = x^2 + y^2$.

30. Consider the following vector fields $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.



- Is $\nabla \cdot \mathbf{F}$ at point A in picture (a) 0, positive, or negative?
- Is $\nabla \cdot \mathbf{F}$ at point B in picture (b) 0, positive, or negative?
- Is $\nabla \cdot \mathbf{F}$ at point C in picture (b) 0, positive, or negative?
- Is $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ at point D in picture (c), 0, positive, or negative?
- Is $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ at point E in picture (d), 0, positive, or negative?

31. Consider the vector field
$$\mathbf{F}(x,y,z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle.$$
Evaluate the work done by $\mathbf{F}$ in moving a particle from point $(0,0,0)$ to point $(1,1,1)$ along each of the following paths.

- C_1 is the straight line segment from $(0,0,0)$ to $(1,1,1)$.
- C_2 consists of three line segments, the first from $(0,0,0)$ to $(1,0,0)$, the second from $(1,0,0)$ to $(1,1,0)$, and the third from $(1,1,0)$ to $(1,1,1)$.
- C_3 is the curve $\langle t, t^2, t^3 \rangle$, $0 \le t \le 1$.
- C_4 it the curve $\langle t\sin(\frac{\pi}{2}t^2), te^{t^2-1}, t^3 \rangle$, $0 \le t \le 1$.

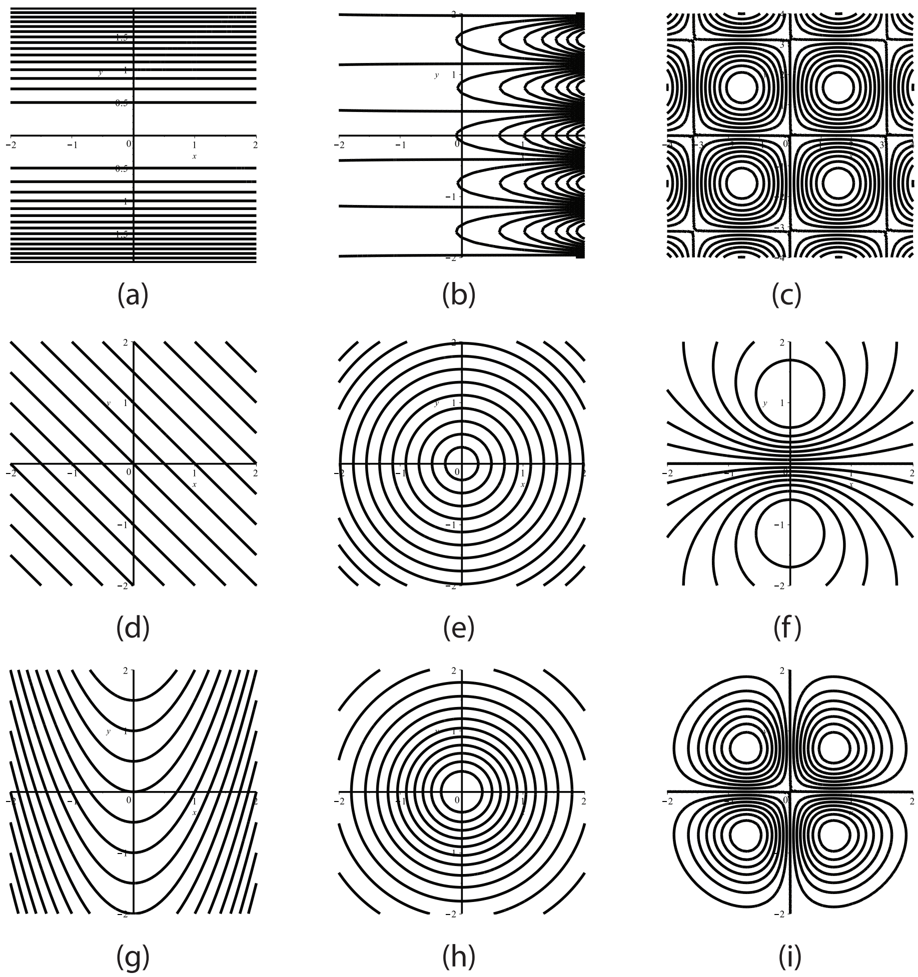
32. Evaluate the surface area of the part of the surface $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$.

33. Find the volume of the region that lies above the paraboloid $z = 2x^2 + 2y^2$ and lies below the cone $z = 2\sqrt{x^2+y^2}$.

34. Consider the sphere $x^2 + y^2 + z^2 = 9$ which models an imaginary planet. Suppose that the temperature on this sphere is given by the function
$$T(x,y,z) = 2x + 2y + z.$$
Find the largest temperature and the smallest temperature on this sphere.

35. Evaluate the integral
$$\int_0^1 \int_0^1 f(x,y)\, dxdy,$$
where
$$f(x,y) = \begin{cases} y & \text{if } y > x^2 \\ x^2 & \text{if } y \le x^2 \end{cases}.$$

36. Consider the following level curves. Answer the following questions.



- Which picture represents the level curves of $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$.
- Which picture represents the level curves of $f(x, y) = x + y$.
- Which picture represents the level curves of $f(x, y) = y^2$.
- Which picture represents the level curves of $f(x, y) = \sin(x)\sin(y)$.
- Which picture represents the level curves of $f(x, y) = y - x^2$.

37. Let $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Let P be the plane $x + y + z = 7$. Write the vector $\mathbf{A}$ as a sum

$$\mathbf{A} = \mathbf{B} + \mathbf{C}$$

where $\mathbf{B}$ is perpendicular to the plane P and $\mathbf{C}$ is parallel to the plane P.

38. Consider the curve given by the equation $x = t$, $y = 2\cos(t)$, and $z = 2\sin(t)$. There is one point A on this curve with $x = 1$ and one point B on this curve with $x = 1$. Consider the part of the curve from point A to point B. Find the length of it.

39. Find the equation of the plane that contains the point $(1, 1, 1)$ and the line $x = t$, $y = 2t + 1$, $z = 3t + 2$.

40. Consider the plane P defined by $x - 2y + 2x = 5$ and the line L defined by $r(t) = \langle 2t - 4, 2t + 1, t + 1\rangle$. Prove that P and L are parallel to each other. Find the distance between P and L.

41. Consider two surfaces, S_1 defined by $z = x^3 + y^4$ and S_2 defined by $z = x^2y^2 + xy$. The intersection of these two surfaces is a curve C and C contains the point $(1, 1, 2)$. Find the tangent line to the curve C at point $(1, 1, 2)$.

42. Evaluate the integral

$$\iiint_E \frac{2}{\sqrt{x^2 + y^2 + z^2}} e^{x^2+y^2+z^2} dV$$

where E is the portion of the solid ball of radius 1 centered at the origin in the first octant.

43. Let

$$\mathbf{F} = (\cos(z) + xy^2)\mathbf{i} + xe^{-z}\mathbf{j} + (\sin(y) + x^2z)\mathbf{k}.$$

Let E be the solid region bounded by the surface $z = x^2 + y^2$ and the surface $z = 8 - x^2 - y^2$. Let S be the boundary surface of E. Evaluate the flux of $\mathbf{F}$ across S.

44. Consider the surface $z = x^2y^2$. The cylinder $x^2 + y^2 = 1$ divides this surface into two parts, one of finite size and the other of infinite size. Let S be the part of finite size. Evaluate

$$\iint_S \langle x, 0, z\rangle \cdot d\mathbf{S}.$$

45. Consider the ball B of radius R centered at the origin. The cone with opening angle a, given by the equation $\varphi = a$, divides the ball into two solids. Let B_a be the solid containing $(0, 0, R)$. Evaluate the ratio of the volume of B_a to the volume of B. Evaluate the ratio of the surface area of B_a to the surface area of B.

46. Let C be the curve of intersection of the plane $xz = 2$ and the cylinder $x^2 + y^2 = 1$. The curve is oriented counterclockwise when viewed from the above. Let

$$\mathbf{F}(x, y, z) = \langle -y + e^{-x^2}, x^2, -z^3\rangle.$$

Evaluate the circulation of $\mathbf{F}$ along C.

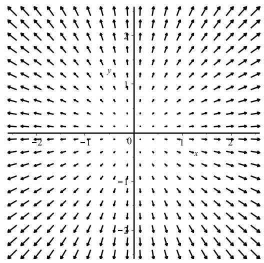
47. Find a parametric equation for the line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$.

48. Evaluate the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

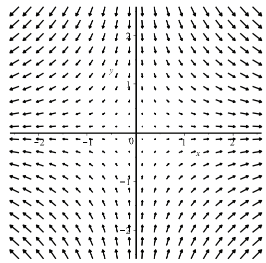
49. Find the extreme values of $f(x, y) = 4x + 2y + 1$ on the disc $x^2 + y^2 \leq 1$.

50. Let D be the region that lies inside the circle $x^2+y^2 = 2y$ but lies outside the circle $x^2+y^2 = 1$. Find the area of D.

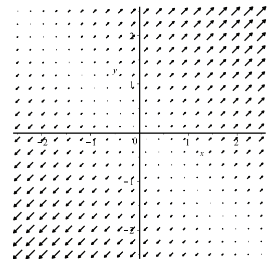
51. Consider the following vector fields:



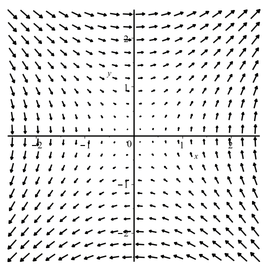
(a)



(b)



(c)



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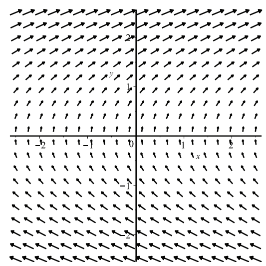

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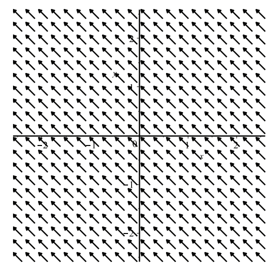

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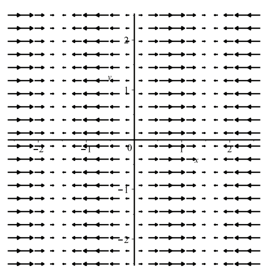

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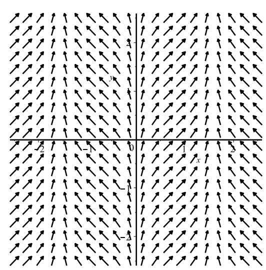

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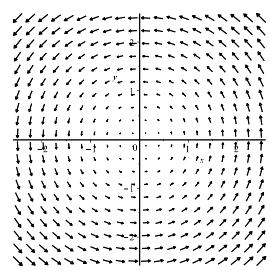

(i)

- Which picture represents the vector field $\langle -1, 1\rangle$?
- Which picture represents the vector field $\langle x, -y\rangle$?
- Which picture represents the vector field $\langle -y, x\rangle$?
- Which picture represents the gradient field of $f(x, y) = xy$?

52. Let E be the solid region bounded by the planes $y = 0$, $x = 0$, $z = 1$, and $x + yz = 0$. Evaluate the integral

$$\iiint_E x dV.$$

53. Consider the surface given by the equation $x^2 + y^2 + z^2 = 6$ and another surface given by the equation $z = x^2 + y^2$. The intersection of these two surfaces forms a curve. Find the length of this curve.

54. Two objects travel through space along two different curves. The trajectory of object A is given by the function $r_1(t) = \langle t^2, 7t - 12, t^2 \rangle$, $t \geq 0$, and the trajectory of object B is given by the function $r_2(t) = \langle 4t3, t^2, 5t6 \rangle$, $t \geq 0$.

- Do the objects collide?
- If your answer to is yes, determine which object was traveling at a faster speed at the collision. On the other hand, if your answer to is no, determine whether the curves represented by the trajectories intersect or not.

55. Find an equation of the plane that passes through the point $(1, 2, 1)$ and contains the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$.

56. Evaluate the surface area of the part of the paraboloid $z = x^2 + y^2$ which lies inside the cylinder $x^2 + y^2 = 9$.

57. Find the largest volume of a rectangular box with sides parallel to the coordinate planes that is contained in the solid ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$, where $a, b, c > 0$.

58. Evaluate the integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{3(x^2+y^2)}}^{\sqrt{4-x^2-y^2}} \sqrt{x^2 + y^2 + z^2} dzdydz.$$

59. Suppose that the electric field in a charged plasma at some instant of time is given by $\mathbf{E}(x, y, z) = \langle 1 - 2x, 2 + 3y, 1 + z \rangle$. According to Gauss's Law, the total electric charge contained within a closed surface S is proportional to the outward flux of the electric field across S. Let S be the surface whose sides S_1 are given by the piece of the paraboloid $x^2 + y^2 = z$ for $0 \leq z \leq 4$, and whose top S_2 is the disc of radius 2 which lies in the plane $z = 4$, centered at $(0, 0, 4)$.

- Draw a picture of S, and label S_1 and S_2 clearly.
- Both S_1 and S_2 can be parametrized such that the domain in the parameter space is a disc of radius 2. Give these parametrizations.
- Use the parametrizations from part to write a single area integral over the disc of radius 2, which gives the outward flux of $\mathbf{E}$ across S.
- Now write a single volume integral in cylindrical coordinates which gives the outward flux of $\mathbf{E}$ across S. Do not evaluate this integral.

60. Let a point on Earth close to London, England be expressed by the coordinates (x, y), where x and y are the longitude and latitude, respectively, of the point. Suppose that the temperature of a place close to London is given by the function $T(x, y) = 5 \sin x - 3y + 110$.

- The coordinates of the city of Greenwich, England are $(0, 101/2)$. In which direction should one walk from this point such that the temperature will increase the fastest?
- What is the directional derivative of the function $T(x, y)$ at Greenwich, in the direction towards Aberdeen, Scotland, which has coordinates $(2, 115/2)$?
- Suppose that the coordinates of an African swallow migrating south t hours after it has taken flight are given by $(x(t), y(t))$, where

$$x(t) = (t - 3)^3 \text{ and } y(t) = 52 - \frac{t}{2}.$$

How fast it the temperature changing for the bird when it is directly over Greenwich?