Lecture #1: Parametric Curves: (11.1-11.2)

Functions $F : \mathbb{R} \to \mathbb{R}^2$

Domain and Range.

Example:

$F(t) = (\cos(t), \sin(t)), \ t \geq 0$

$x(t) = \cos(t)$

$y(t) = \sin(t)$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{3\pi}{4}$</td>
<td>$-\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Circle:

$x^2(t) + y^2(t) = \cos^2(t) + \sin^2(t) = 1$.

Example:

Describe the motion of the particles whose coordinates are plotted below.
Example:
Describe the motion of the particle whose \( x, y \) coordinates are given by:
\[
\begin{align*}
x(t) &= \cos(e^{-t^2}), \quad t \geq 0 \\
y(t) &= \sin(e^{-t^2})
\end{align*}
\]
\[x^2 + y^2 = 1 \implies \text{lies on circle.}\]

\[
\begin{array}{c}
\text{Y} \\
\downarrow \\
(\cos(1), \sin(1)) \quad \text{X}
\end{array}
\]

Example:
The path of a particle is given by
\[
\begin{align*}
x &= t^2, \\
y &= \pm t
\end{align*}
\]
Sketch the path of the particle.
\[
t = \pm \sqrt{x}
\]
\[\implies y = \pm \sqrt{x}
\]
\[
\begin{array}{c}
\text{y-axis} \\
\downarrow \\
\text{Not part of graph since } t \geq 0.
\end{array}
\]
\[
\begin{array}{c}
\text{Y} \\
\downarrow \\
(\cos(1), \sin(1)) \quad \text{X}
\end{array}
\]
Motion along a line:
An object moves with constant speed along a line through the point \((x_0, y_0)\). Both the \(x, y\)-coordinates have constant rate of change.

\[ a = \frac{dx}{dt}, \quad \frac{dy}{dt} = b \]

F.T.C.:

\[ x(t) - x(0) = \int_0^t \frac{dx}{ds} ds, \quad y(t) - y(0) = \int_0^t \frac{dy}{ds} ds \]

\[ \Rightarrow x(t) - x_0 = at, \quad y(t) - y_0 = bt \]

\[ \Rightarrow x(t) = at + x_0 \quad y(t) = bt + y_0 \]

Parametric equations for a line.

Slope is given by:

\[ m = \frac{\text{rise}}{\text{run}} = \frac{b}{a} \]

Speed and Velocity:

\[ \mathbf{v}(t) = (x(t), y(t)) \]

\[ \Delta s \sim \sqrt{(\Delta x)^2 + (\Delta y)^2} \]

\[ \Delta t \sim \sqrt{(\Delta x)^2 + (\Delta y)^2} \]
\[ V = \lim_{\Delta t \to 0} \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta t)^2}} \]

\[ = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \]

**Example:**

A particle moves in the xy-plane with

\[ x = 2t^3 - 9t^2 + 12t \]
\[ y = 3t^4 - 16t^3 + 18t^2 \]

where \( t \) is time.

a.) At what times is the particle stopped?

b.) At what times is the particle moving parallel to the x-axis?

\[
\frac{dx}{dt} = 6t^2 - 18t + 12 = 6(t^2 - 3t + 2)
\]
\[
\frac{dy}{dt} = 12t^3 - 36t^2 + 36t = 12t(t^2 - 4t + 3)
\]

\[
\frac{dx}{dt} = 0 \Rightarrow t = 1, 2
\]

\[
\frac{dy}{dt} = 0 \Rightarrow t = 0, 1, 3
\]

Steps moving at \( t = 1 \).
Moving parallel to x-axis \( t = 0, 1, 3 \).
Tangent Lines

Find the tangent line at $(1,2)$ to the curve defined by
\[ x = t^3, \quad y = 2t \]

Two methods:

1. \( \frac{dx}{dt} = 3t^2, \quad \frac{dy}{dt} = 2 \)

\[ \left. \frac{dx}{dt} \right|_{t=1} = 3, \quad \left. \frac{dy}{dt} \right|_{t=1} = 2 \]

The tangent line is:

\( x(t) = 3t + 1 \)

\( y(t) = 2t + 2 \)

2. \[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{3} \]

\[ \Rightarrow y = \frac{3}{2} (x-1) + 2. \]
Concavity
\[ x(t) = f(t), \quad y(t) = g(t) \]

Slope: \[ \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dx}{dt} \] (proof: \[ \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \])

Concavity: \[ \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} \cdot \frac{dt^2}{dx^2} \]

Let \( w = \frac{dx}{dt} \). Then,
\[ \frac{dw}{dx} = \frac{\frac{dx}{dt}}{\frac{dx}{dt}} = \frac{dt}{dx} \cdot \frac{dx}{dt} = \frac{1}{w} \cdot w = 1 \]

Example:
If \( x(t) = \cos(t), \quad y(t) = \sin(1t) \) find the slope and concavity at \( t = \pi/4 \).

\[ \frac{dy}{dx} \bigg|_{t = \pi/4} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{-\sin(t)}{-\cos(t)} \bigg|_{t = \pi/4} = -1 \]

\[ \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{dx}{dt} \cdot \frac{d}{dt} \left( \frac{dx}{dt} \right) \]

\[ = \frac{dx}{dt} \cdot \frac{d}{dt} \left( \frac{\cos(t)}{-\sin(t)} \right) \]

\[ = \frac{dx}{dt} \cdot \frac{-\cos(t)}{-\sin(t)} = \frac{\cos(t)}{\sin(t)} \]

\[ \frac{dx}{dt} \bigg|_{t = \pi/4} = -\frac{1}{\sin^2(t)} \bigg|_{t = \pi/4} = -\frac{1}{(1/2)^2} = -2 \]
Arc Length

Find the circumference of the ellipse

\[ x(t) = 2 \cos(t), \quad y(t) = \sin(t) \]

for \( 0 \leq t \leq 2\pi \).

The quantity:

\[ ds = \sqrt{(dx)^2 + (dy)^2} \, dt \]

is known as the arc-length element.
Surface Area:
Find the surface area of the ellipse
\[ x(t) = 2 \cos(t), \]
\[ y(t) = 3 \sin(t) \]
revolved around

4. The y-axis.
2. The x-axis.

\[ S \approx \sum_{i=1}^{N} 2\pi x(t_i) \Delta s_i \]
\[ \Rightarrow S = \lim_{N \to \infty} \sum_{i=1}^{N} 2\pi x(t_i) \Delta s_i \]
\[ = \pi \int_{-\pi/2}^{\pi/2} 2\pi \cos(t) \sqrt{4 \cos^2(t) + 9 \sin^2(t)} \, dt \]
\[ = 8 \int_{0}^{\pi/2} \cos(t) \sqrt{4 \cos^2(t) + 9 \sin^2(t)} \, dt \]
\[ v = \sin(t), \quad \sin^2(t) + \cos^2(t) = 1 \]
\[ dv = \cos(t) \Rightarrow \cos^2(t) = 1 - v^2 \]

\[ \Rightarrow S = 8 \int_{0}^{1} \sqrt{4 - 3v^2} \, dv \]
\[ = 8 \left( \frac{1}{2} + \frac{2}{9} \sqrt{3} \pi \right) \]
About the $y$-axis

$$S = 2 \int_0^{\pi/2} \sin(t) \sqrt{4 \cos^2(t) + \sin^2(t)} \, dt$$

Let $u = \cos(t)$, so $du = -\sin(t) \, dt$

$$S = -2 \int_1^0 \sqrt{4u^2 + 1 - u^2} \, du$$

$$= 2 \int_0^1 \sqrt{3u^2 + 1} \, du$$

$$= \frac{1}{3} \left( 6 + \sqrt{3} \sinh^{-1}(\sqrt{3}) \right)$$