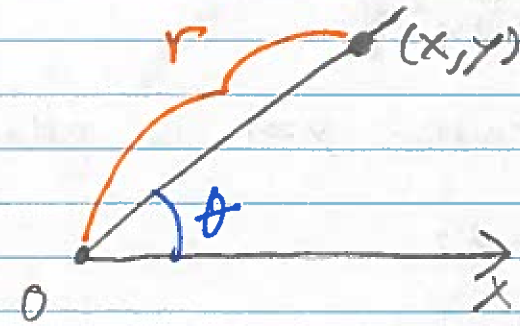
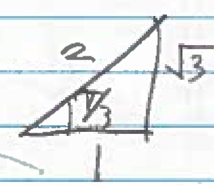
11.3-11.5: Polar Coordinates



$$x = r\cos\theta, \quad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}.$$

Example:

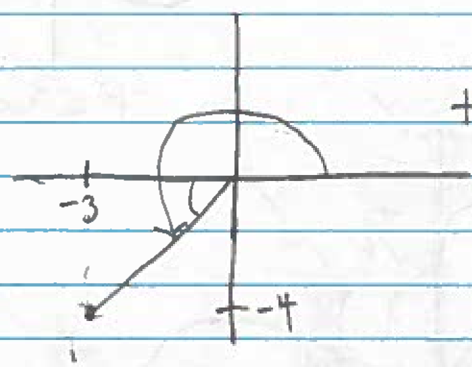
1. Give Cartesian coordinates for the point with polar coordinate $r = 7$, $\theta = \pi/3$



$$\Rightarrow x = 7\cos(\pi/3) = \frac{1}{2}$$
$$y = 7\sin(\pi/3) = \frac{\sqrt{3}}{2}$$

2. Convert to polar coordinates $x = -3$, $y = -4$.

$$r^2 = 25 \Rightarrow r = 5$$



$$\tan^{-1}\left(\frac{-4}{-3}\right) = .927$$
$$\Rightarrow \theta = \pi + \tan^{-1}\left(\frac{4}{3}\right)$$

example:

Write the equation $r=1$ in Cartesian coordinates:

$$r=1 \Rightarrow x^2+y^2=1$$

Write the equation $y=1$ in polar coordinates

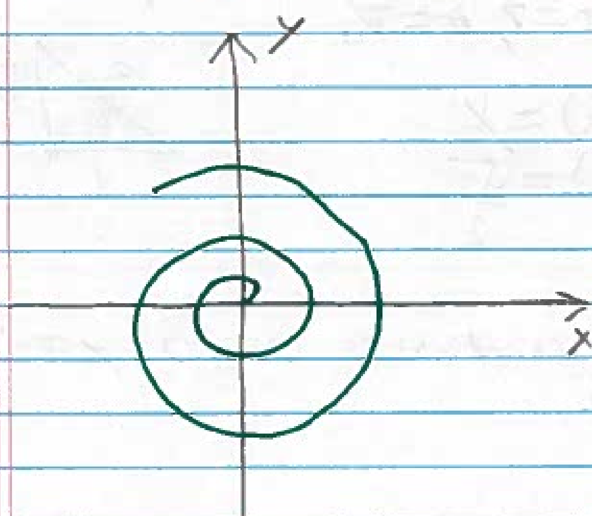
$$1 = r\cdot\sin\theta$$

$$\Rightarrow r = \frac{1}{\sin\theta}$$

example:

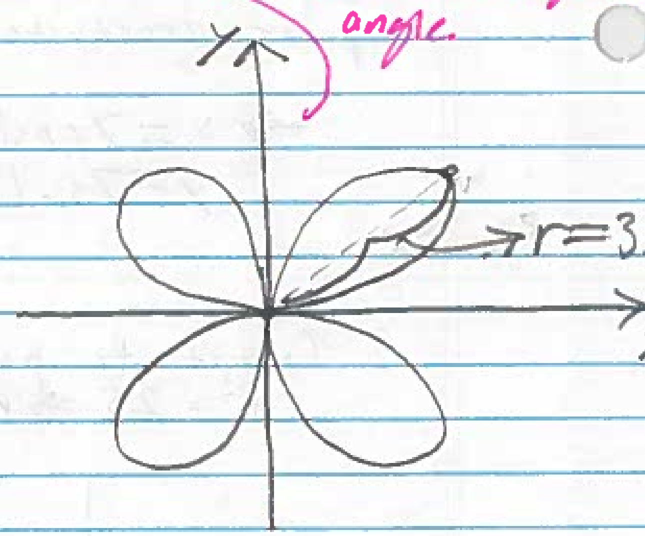
Sketch the following curves defined in polar coordinates

1. $r=\theta$
2. $r=3\sin 2\theta$
3. $r=4\cos 3\theta$

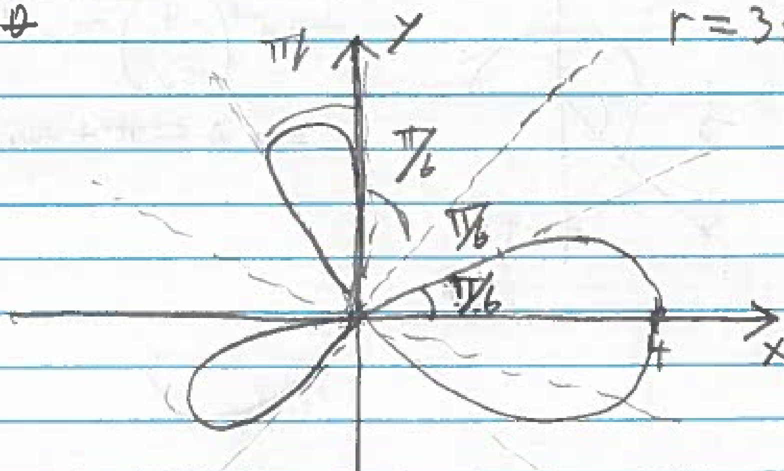


$r=\theta$

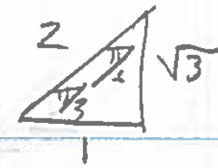
If $r<0$ then use negative angle.



$r=3\sin 2\theta$



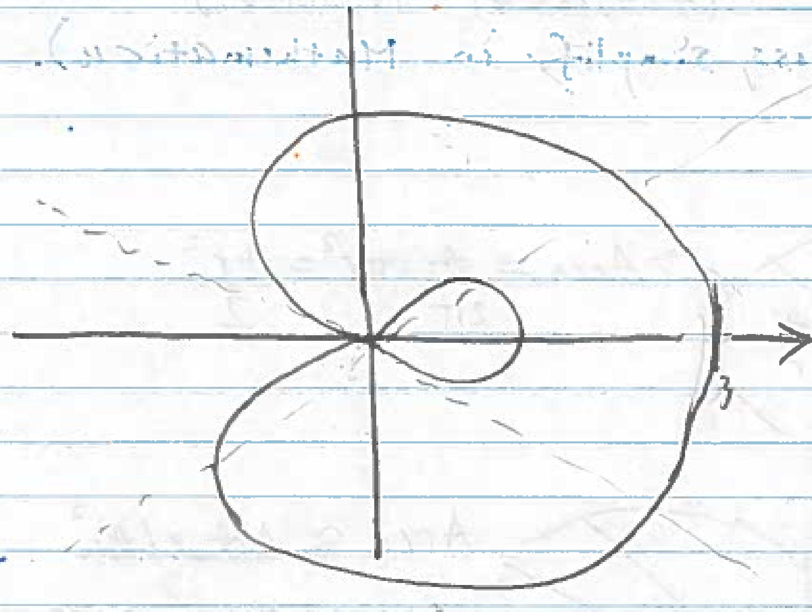
$r=4\cos(3\theta)$



<u>example:</u>

Sketch a graph of:

$r = 1 + 2\cos\theta$, $r = 0$ when $\theta =$



<u>Slope and Curvature:</u>

$r = f(\theta)$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\frac{d}{d\theta}\left(f(\theta)\sin(\theta)\right)}{\frac{d}{d\theta}\left(f(\theta)\cos(\theta)\right)}$$

$$= \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin\theta}$$

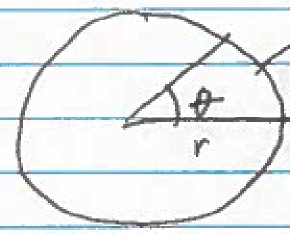
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{f'(\theta)\sin\theta + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}\right)$$

$$= \frac{d\theta}{dx}\frac{d}{d\theta}\left(\frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}\right)$$

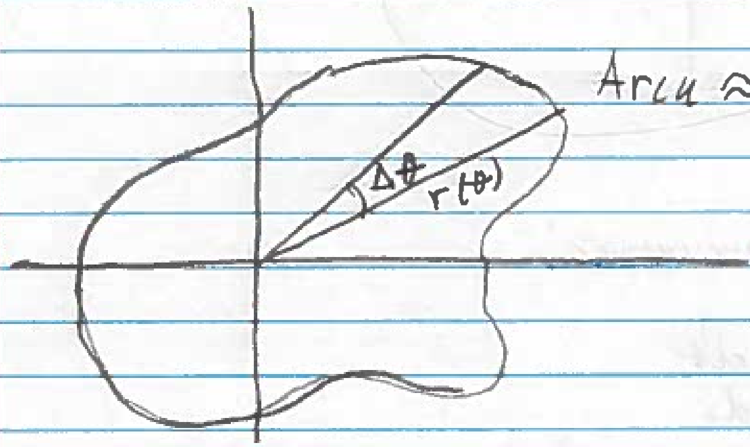
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\left[(f'(\theta)\cos(\theta) - f(\theta)\sin(\theta))\cdot(f''(\theta)\sin(\theta) + f'(\theta)\sin\theta + f'(\theta)\cos(\theta) - f(\theta)\sin\theta) - (f'(\theta)\sin(\theta) + f(\theta)\cos(\theta))\cdot(f''(\theta)\cos(\theta) - f'(\theta)\sin(\theta) - f'(\theta)\sin(\theta) - f(\theta)\cos(\theta))\right]}{(f'(\theta)\cos(\theta) - f(\theta)\sin(\theta))^3}$$

(gross, simplify in Mathematica).

Area:



$$\text{Area} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{\theta r^2}{2}.$$



$$\text{Area} \approx \frac{\Delta\theta\, r(\theta)^2}{2}$$

$$\Rightarrow \text{Total area} = \int_0^{2\pi} \frac{r(\theta)^2}{2}\, d\theta$$

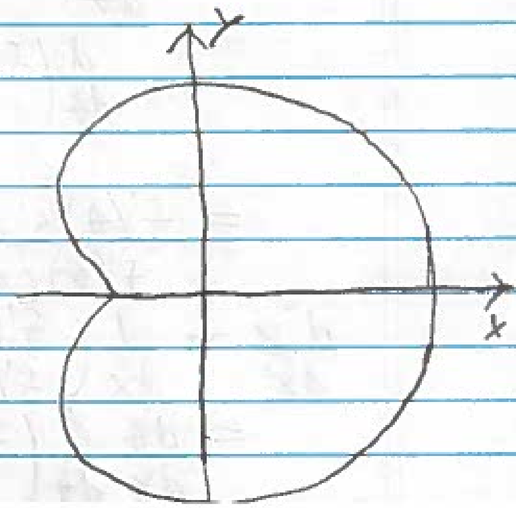
example:

Find area of:

$r(\theta) = 3 + 2\cos(\theta)$

$$\text{Area} = \int_0^{2\pi} \frac{(3+2\cos(\theta))^2}{2}\, d\theta$$

$= 11\pi.$



Arclength

$x = f(\theta)\cos(\theta)$

$y = f(\theta)\sin(\theta)$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{\left(f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)\right)^2 + \left(f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)\right)^2}\, d\theta$$

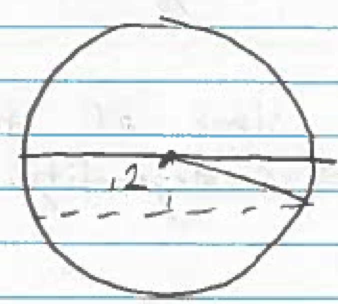
$$= \int_\alpha^\beta \sqrt{(f'(\theta))^2\cos^2\theta + f(\theta)\sin^2\theta + f'(\theta)\sin^2\theta + f(\theta)\cos^2(\theta)}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{df}{d\theta}\right)^2 + f(\theta)^2}\, d\theta.$$

Example:

A cylindrical oil tank of radius 2 ft is lying on its side. If the oil is 1.8 ft deep, what percentage of a full tank is left.

Interesting problem. Have students do it.



$y = -.2$

$\Rightarrow r\sin\theta = -.2$

$\Rightarrow r = \dfrac{-.2}{\sin(\theta)}$

$A =$

Example:

For $r = \sin(3\theta)$, ~~find locations of all horizontal and vertical tangent lines.~~ At the points where $|r|$ is maximized, show that the tangent line is orthogonal to the line connecting the point to the origin.

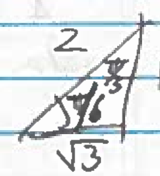
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(\sin(3\theta)\sin(\theta))}{\frac{d}{d\theta}(\sin(3\theta)\cos(\theta))}$$

$$= \frac{3\cos(3\theta)\sin(\theta) + \sin(3\theta)\cos(\theta)}{3\cos(3\theta)\cos(\theta) - \sin(3\theta)\sin(\theta)}$$

$$\frac{dy}{dx} = 0 \Rightarrow 3\cos(3\theta)\sin(\theta) + \sin(3\theta)\cos(\theta) = 0.$$

$$\frac{dr}{d\theta} = 3\cos(3\theta) = 0 \Rightarrow \theta = \pi/6$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{\sin(\pi/2)\cos(\pi/6)}{-\sin(\pi/2)\sin(\pi/6)} = -\cot(\pi/6) = -\sqrt{3}.$$



The slope of the radial line is $1/\sqrt{3}$
$\Rightarrow$ orthogonality.