1 Problems for everybody

1. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$.
   a) Prove that $\langle \vec{u} + \vec{v}, \vec{u} - \vec{v} \rangle = \| \vec{u} \|^2 - \| \vec{v} \|^2$.
   b) Prove that if $\vec{u}$ and $\vec{v}$ have the same norm, then $\vec{u} + \vec{v}$ is orthogonal to $\vec{u} - \vec{v}$.
   c) Prove that the diagonals of a rhombus are orthogonal to each other.

2. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Prove the parallelogram law:

   $\| \vec{u} + \vec{v} \|^2 + \| \vec{u} - \vec{v} \|^2 = 2 (\| \vec{u} \|^2 + \| \vec{v} \|^2)$.

3. Suppose $A \in \mathbb{R}^{n \times n}$ satisfies $\| A \|_2 \leq 1$. Prove that $A - \sqrt{2} I$ is invertible.

4. Let $1 \leq p \leq \infty$. Suppose $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix with diagonal entries $d_1, \ldots, d_n \in \mathbb{R}$. Prove that

   $\| D \|_p = \max_{1 \leq i \leq n} |d_i|.$

   **Hint**: One way to do this problem is to show that $\max_{1 \leq i \leq n} |d_i|$ is both an upper and lower bound for $\| D \|_p$.

5. Problems 2.1, 2.2, 2.6, 3.2, 3.3

2 Problems for mathematics/statistics graduate students

1. Problems 2.3, 2.4, 2.5