Homework 2

Numerical Linear Algebra

September 9, 2017

1 Problems for everybody

1. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$.

   a) Prove that $\langle \vec{u} + \vec{v}, \vec{u} - \vec{v} \rangle = \|\vec{u}\|^2 - \|\vec{v}\|^2$.

   b) Prove that if $\vec{u}$ and $\vec{v}$ have the same norm, then $\vec{u} + \vec{v}$ is orthogonal to $\vec{u} - \vec{v}$.

   c) Prove that the diagonals of a rhombus are orthogonal to each other.

2. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Prove the **parallelogram law:**

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right).$$

3. Suppose $A \in \mathbb{R}^{n \times n}$ satisfies $\|A\|_2 \leq 1$. Prove that $A - \sqrt{2}I$ is invertible.

4. Let $1 \leq p \leq \infty$. Suppose $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix with diagonal entries $d_1, \ldots, d_n \in \mathbb{R}$. Prove that

$$\|D\|_p = \max_{1 \leq i \leq n} |d_i|.$$

   Hint: One way to do this problem is to show that $\max_{1 \leq i \leq n} |d_i|$ is both an upper *and* lower bound for $\|D\|_p$.

5. Problems 2.1, 2.2, 2.6, 3.2, 3.3

2 Problems for mathematics/statistics graduate students

1. Problems 2.3, 2.4, 2.5