#27.3 Homework #10

In summation notation:

$$r(\vec{x}) = \frac{\sum_{i=1}^{n}\sum_{k=1}^{n} x_i A_{ik} x_k}{\sum_{\ell=1}^{n} x_\ell^2}.$$

Therefore,

$$\frac{\partial}{\partial x_j} r(\vec{x}) = \frac{\sum_{\ell=1}^{n} x_\ell^2\left(\sum_{i=1}^{n}\sum_{k=1}^{n}\left(\delta_{ij}A_{ik}x_k + x_i A_{ik}\delta_{kj}\right)\right) - \sum_{i=1}^{n}\sum_{k=1}^{n} x_i A_{ik} x_k \sum_{\ell=1}^{n} 2x_\ell \delta_{\ell j}}{\left(\sum_{\ell=1}^{n} x_\ell^2\right)^2}$$

$$= \frac{\sum_{\ell=1}^{n}\sum_{k=1}^{n} x_\ell^2 A_{jk} x_k + \sum_{\ell=1}^{n}\sum_{i=1}^{n} x_\ell^2 x_i A_{ij} - \sum_{i=1}^{n}\sum_{k=1}^{n} x_i A_{ik} x_k \cdot 2x_j}{\left(\sum_{\ell=1}^{n} x_\ell^2\right)^2}$$

$$= \frac{\sum_{\ell=1}^{n} x_\ell^2\left(\sum_{k=1}^{n}\left(A_{jk}x_k + x_i A_{kj}\right)\right) - 2x_j\sum_{i=1}^{n}\sum_{k=1}^{n} x_i A_{ik} x_k}{\left(\sum_{\ell=1}^{n} x_\ell^2\right)^2}$$

$$= \frac{(A\vec{x})_j + (A^T\vec{x})_j - (2r(\vec{x})\vec{x})_j}{\left(\sum_{\ell=1}^{n} x_\ell^2\right)}$$

$$\Rightarrow \nabla r(\vec{x}) = \frac{A\vec{x} + (A^T\vec{x}) - 2r(\vec{x})\vec{x}}{\vec{x}^T\vec{x}}.$$

There, if $\vec{v}$ is an eigenvector it follows that

$$\nabla r(\vec{v}) \neq 0.$$

Therefore, Taylor expanding it follows that

$$r(\vec{x}) = r(\vec{v}) + \nabla r(\vec{v})\cdot(\vec{x}-\vec{v}) + \dots$$

$$\Rightarrow |r(\vec{x}) - \lambda| \leq \|\nabla r(\vec{v})\|\cdot\|\vec{x}-\vec{v}\| + \dots$$

$$\Rightarrow |r(\vec{x}) - \lambda| = O(\|\vec{x}-\vec{v}\|)$$

This implies for non-Hermitian matrices that Rayleigh quotient iteration is quadratic.

#27.6

Generically, if two eigenvalues are equal their geometric multiplicity is 2. Consequently, the stationary points of $r(\vec{x})$ form a great circle on the sphere.

$\blacksquare$

#24.2

a.) Let $\vec{v}$ be an eigenvector with corresponding eigenvalue λ. Let $1 \le j \le n$ satisfy $|v_j| = \max_{1 \le i \le n} |v_i|$. Therefore,

$$\lambda v_j = \sum_{i=1}^{n} a_{ij} v_j$$

$$\Rightarrow (\lambda - a_{jj}) v_j = \sum_{i \ne j}^{n} a_{ij} v_i$$

$$\Rightarrow (\lambda - a_{jj}) = \sum_{i \ne j}^{n} a_{ij} \frac{v_i}{v_j}$$

$$\Rightarrow |\lambda - a_{jj}| \le \sum_{i \ne j}^{n} |a_{ij}|$$

$\blacksquare$