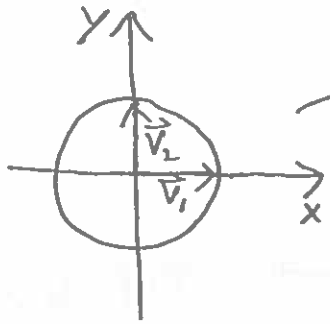
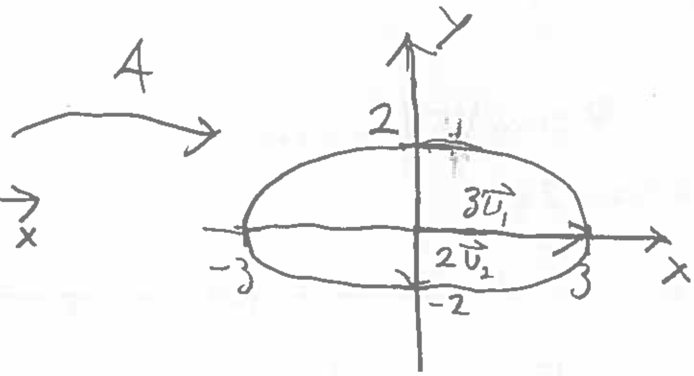
Homework #3.

#1.

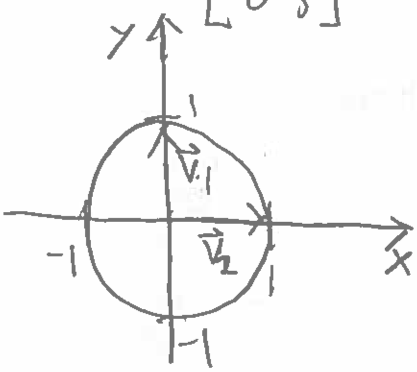
a.) $A=\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}$

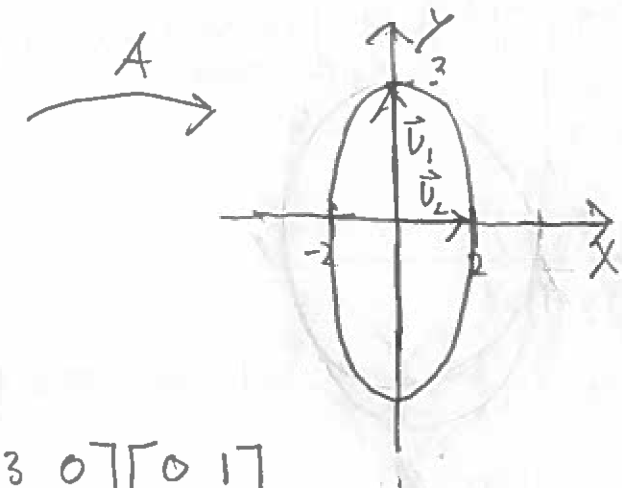




$$\Rightarrow A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

b.) $B=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$





$$\Rightarrow B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

c.). $C=\begin{bmatrix}1&1\\0&0\end{bmatrix}$

Clearly $range(C)=span\{\begin{bmatrix}1\\0\end{bmatrix}\}$. Let $\begin{bmatrix}\cos(\theta)\\\sin(\theta)\end{bmatrix}=\vec{x}$. Then,

$$C\vec{x}=\begin{bmatrix}\cos\theta+\sin\theta\\0\end{bmatrix}$$

$$\Rightarrow \|C\vec{x}\|_2^2=1+2\cos(\theta)\sin(\theta)=1+\sin(2\theta),$$

$$\Rightarrow \frac{d}{d\theta}\|C\vec{x}\|_2^2=2\cos(2\theta).$$

Therefore, $\|C\vec{x}\|_2^2$ obtains its maximum value at $\theta=\pi/4$. Therefore,

$$C=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\sqrt{2}&0\\0&0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\-1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}$$

d.). $D=\begin{bmatrix}1&1\\1&1\end{bmatrix}$.

Clearly, $range(D)=span\{\begin{bmatrix}1\\1\end{bmatrix}\}$. Let $\begin{bmatrix}\cos(\theta)\\\sin(\theta)\end{bmatrix}=\vec{x}$. Then,

$$D\vec{x}=\begin{bmatrix}\cos\theta+\sin\theta\\\cos\theta+\sin\theta\end{bmatrix}$$

$$\Rightarrow \|D\vec{x}\|_2^2=2+2\sin(2\theta)$$

$$\Rightarrow \frac{d}{d\theta}\|D\vec{x}\|_2^2=4\cos(2\theta)$$

Therefore, $\|D\vec{x}\|_2^2$ obtains its maximum value at $\theta=\pi/4$. Therefore,

$$D=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}\begin{bmatrix}2&0\\0&0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}$$

#2.

Let $A=\begin{bmatrix}-2 & 11\\ -11 & 5\end{bmatrix}$ and $\vec{x}=\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$. Then,

$$\|A\vec{x}\|_2^2=(-2\cos\theta+11\sin\theta)^2+(-11\cos\theta+5\sin\theta)^2$$
$$=4\cos^2\theta-44\cos\theta\sin\theta+11^2\sin^2\theta+11^2\cos^2\theta-2\cdot 55\cos\theta\sin\theta+25\sin^2\theta$$
$$=11^2+4+21\sin^2\theta-154\cos\theta\sin(\theta)$$
$$=11^2+4+21\sin^2\theta-154(1-\sin^2\theta)^{1/2}\sin(\theta)$$

$$\Rightarrow \frac{d}{d\theta}\|A\vec{x}\|_2^2=42\sin\theta\cos\theta+154\sin^2\theta-154\cos^2\theta$$
$$=21\sin(2\theta)-154\cos(2\theta)$$

Therefore, the critical value occurs at:

$$\theta=\frac{1}{2}\tan^{-1}\left(\frac{154}{21}\right).$$

$$\Rightarrow x=\cos\left(\frac{1}{2}\tan^{-1}\left(\frac{154}{21}\right)\right)\quad,\quad y=\sin\left(\frac{1}{2}\tan^{-1}\left(\frac{154}{21}\right)\right)$$

$$=\left(\sqrt{\frac{1+\cos\left(\tan^{-1}\left(\frac{154}{21}\right)\right)}{2}}\right)\quad,\quad =\left(\sqrt{\frac{1-\cos\left(\tan^{-1}\left(\frac{154}{21}\right)\right)}{2}}\right)$$

$$=\left(\sqrt{\frac{1+\frac{21}{\sqrt{21^2+154^2}}}{2}}\right)\quad,\quad =\left(\sqrt{\frac{1-\frac{21}{\sqrt{21^2+154^2}}}{2}}\right)$$

$$=x^*\quad\quad =y^*$$

Using $\vec{x}=\begin{bmatrix}x^*\\ y^*\end{bmatrix}$, the rest of the calculation is trivial.

#5.2

Let $A \in \mathbb{R}^{n \times n}$ with SVD $A = U \Sigma V^*$. Let

$$A_n = U(\Sigma + \tfrac{1}{n} I) V^*$$

Then,

$$\begin{aligned} \|A - A_n\|_2 &= \| U \Sigma V^* - U(\Sigma + \tfrac{1}{n} I) V^* \|_2 \\ &= \| U (\tfrac{1}{n} I) V^* \|_2 \\ &\leq \|U\|_2 \cdot \tfrac{1}{n} \cdot \|V^*\|_2 \\ &= \tfrac{1}{n}. \end{aligned}$$

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#5.4

Let $\vec{v} = \begin{bmatrix} \vec{x}_1 \\ \hline \vec{x}_n \end{bmatrix}$ be an eigenvector of $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$ with corresponding eigenvalue λ. Then,

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \end{bmatrix} = \begin{bmatrix} A^* \vec{x}_2 \\ A \vec{x}_1 \end{bmatrix} = \begin{bmatrix} \lambda \vec{x}_1 \\ \lambda \vec{x}_2 \end{bmatrix}.$$

$$\Longrightarrow \begin{aligned} A A^* \vec{x}_2 &= \lambda A \vec{x}_1 = \lambda^2 \vec{x}_2 \\ A^* A \vec{x}_1 &= \lambda A^* \vec{x}_2 = \lambda^2 \vec{x}_1. \end{aligned}$$

Letting $A = U \Sigma V^*$ it follows that:

$$\begin{aligned} U \Sigma V^* \cdot V \Sigma U^* \vec{x}_2 &= U \Sigma \cdot \Sigma U^* \vec{x}_2 = \lambda^2 \vec{x}_2. \\ V \Sigma U^* U \Sigma V^* \vec{x}_1 &= V \Sigma \Sigma V^* \vec{x}_1 = \lambda^2 \vec{x}_1. \end{aligned}$$

Thus the eigenvectors are simply

$$\vec{v} = \begin{bmatrix} \vec{v}_i \\ \vec{u}_i \end{bmatrix},$$

with eigenvalues σ_i^2, where $\vec{v}_i, \vec{u}_i$ are the right and left singular vectors respectively.

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