Homework #4,

#6.1

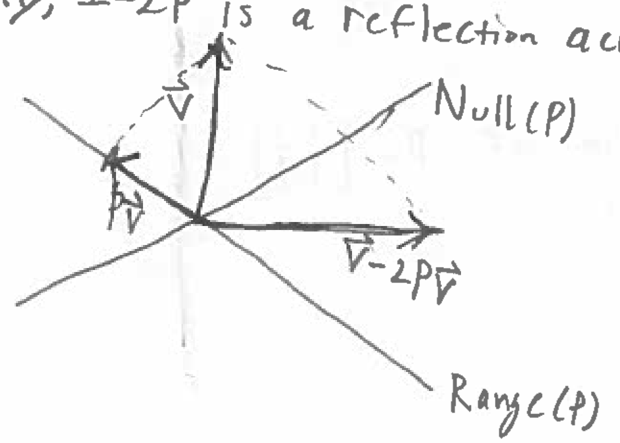
If P is an orthogonal projector, then I−2P is unitary.

proof:

$$(I-2P)^*(I-2P) = (I-2P^*)(I-2P)$$
$$= I - 2P^* - 2P + 4P^*P$$
$$= I - 4P + 4P$$
$$= I.$$

Therefore, $(I-2P)^* = (I-2P)^{-1}$!

Geometrically, I−2P is a reflection across the nullspace of P:



#6.4

Let $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, find the orthogonal projection onto range(A).

Solution:

I am going to do this by Gram-Schmidt.

$$\vec{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \Rightarrow r_{11} = \sqrt{2}$$

$$\vec{a}_2 = r_{12}\vec{q}_1 + r_{22}\vec{q}_2$$

$$r_{12} = \vec{q}_1^{\,T} \cdot \vec{a}_2 = 2/\sqrt{2} = \sqrt{2}$$

$$\Rightarrow \vec{q}_2' = \vec{a}_2 - r_{12}\vec{q}_1$$
$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \vec{q}_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \quad r_{22} = \sqrt{3}.$$

Therefore, the projection matrix is given by:

$$P = QQ^*$$
$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$
$$= \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix}$$

#7.1

Determine a QR factorization of $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution:

From our previous calculation:

$$B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

#2

Let $A \in \mathbb{R}^{n\times n}$, prove the following:

i. $null(A^T) = [range(A)]^\perp$
ii $range(A^T) = [null(A)]^\perp$
iii $null(A) = [range(A^T)]^\perp$
iv. $range(A) = [null(A^T)]^\perp$

Solution:

i.) Let $\vec{x} \in null(A^T)$ and $\vec{b} \in range(A)$. Since $\vec{b} \in range(A)$ there exists $(\Rightarrow)$ $\vec{y} \in \mathbb{R}^n$ such that $A\vec{y} = \vec{b}$. Consequently,

$$\vec{b}^T\vec{x} = (A^T\vec{y}^T)\vec{x} = \vec{y}^T A^T \vec{x} = 0$$
$$\Rightarrow \vec{x} \in [range(A)]^\perp.$$

$(\Leftarrow)$ Suppose $\vec{x} \in [range(A)]^\perp$. Then, for all $\vec{b} \in range(A)$

$$\vec{b}^T\vec{x} = 0$$

Therefore, for all $\vec{y} \in \mathbb{R}^n$;

$$(A\vec{y})^T \vec{x} = 0.$$

$$\Longrightarrow \vec{y}^T A^T \vec{x} = 0.$$

Consequently, $A^T\vec{x}$ is orthogonal to $\mathbb{R}^n$. Therefore, $A^T\vec{x} = 0$. ∎

#6.5.

Let $P \in \mathbb{R}^{m \times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with equality if and only if P is an orthogonal projector.

proof:

$$\|P\| = \|P^2\| \leq \|P\| \cdot \|P\|$$

$$\Longrightarrow \|P\| \geq 1.$$

If P is an orthogonal projector than $\|P\| = 1$ since the SVD of P is of the form

$$P = Q\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Q^*.$$

Now, suppose P is a projection with $\|P\| = 1$. Then,

$$\begin{aligned}
\langle P\vec{x}, \vec{x} - P\vec{x} \rangle &= (P\vec{x})^T \cdot \vec{x} - (P\vec{x})^T \cdot (P\vec{x}), \\
&= x^T P^T \vec{x} - x^T P^T P \vec{x}, \\
&\leq \|x\| \cdot \|Px\| - \|Px\|^2, \\
&= \|\vec{x}\| \cdot \|P\vec{x}\| - 1, \\
&\leq \|P\|_2 - 1, \\
&= 0.
\end{aligned}$$