Homework #8

#12.1

Suppose $A \in \mathbb{R}^{202 \times 202}$ with $\|A\|_2 = 100$ and $\|A\|_F = 101$. Give the sharpest possible lower bound on the 2-norm condition number $K(A)$.

Solution:

Since $\|A\|_2 = 100$ and $\|A_F\| = 101$ it follows that the singular values of A satisfy:

$$\sigma_1 = 100, \quad \sigma_1^2 + \dots + \sigma_{202}^2 = 101^2, \quad \sigma_1 \geq \dots \geq \sigma_{202}$$

Therefore,

$$\sigma_2^2 + \dots + \sigma_{202}^2 = 201$$

$$\Rightarrow 201 \geq 201\,\sigma_{202}^2$$

$$\Rightarrow \frac{1}{\sigma_{202}} \geq 1$$

Therefore,

$$K(A) = \frac{\sigma_1}{\sigma_{202}} \geq \frac{100}{1} = 100.$$

#14.1

a. True
b. True
c.) True.
e.) True.

#14.2

Show that:

a.) $(1+O(\varepsilon_m))(1+O(\varepsilon_m)) = 1 + O(\varepsilon_m)$

b.) $(1+O(\varepsilon_m))^{-1} = 1 + O(\varepsilon_m)$.

Solution:

If $f = (1+O(\varepsilon_m))(1+O(\varepsilon_m))$ then

$$f \leq (1+k_1\varepsilon_m)(1+k_2\varepsilon_m) = 1 + k_1\varepsilon_m + k_2\varepsilon_m + k_1k_2\varepsilon_m^2 \leq 1 + (k_1+k_2+k_1k_2)\varepsilon_m$$

$$\Rightarrow f = 1 + O(\varepsilon_m).$$

If $f=(1+O(\varepsilon_m))^{-1}$ then

$$f \le \frac{1}{1+k_1\varepsilon}$$

$$= 1-k_1\varepsilon+k_1^2\varepsilon^2-k_1^3\varepsilon^3+\dots$$

$$\le 1-k_1\varepsilon+D\varepsilon^2$$

$$\rightarrow |f| \le 1+|k_1|\varepsilon+|D|\varepsilon^2$$

$$\le 1+|k_1|\varepsilon+|D|\varepsilon$$

$$\Rightarrow |f| = 1+O(\varepsilon_m)$$

■

#15.1

a.) Data $x\in\mathbb{R}$, $f(x)=x^2$.

Solution:

$$\tilde{f}(x) = fl(x)\otimes fl(x)$$

$$= x(1+\varepsilon_1)\otimes(1+\varepsilon_1)\cdot x$$

$$= x^2(1+\varepsilon_1)^2(1+\varepsilon_2)$$

$$= x^2(1+\varepsilon_1)^2\sqrt{1+\varepsilon_2}^{\,2}$$

$$= f(x(1+\varepsilon_1)\sqrt{1+\varepsilon_2})$$

$$= f(\tilde{x}).$$

Now,

$$\frac{|\tilde{x}-x|}{|x|} = \frac{|x(1+\varepsilon_1)\sqrt{1+\varepsilon_2}-x|}{|x|}$$

$$= \frac{|x(1+\varepsilon_1)(1+\frac{1}{2}\varepsilon_2+D\varepsilon_2^2)-x|}{|x|}$$

$$= |\varepsilon_1+\tfrac{1}{2}\varepsilon_2+D\varepsilon_2^2+\tfrac{1}{2}\varepsilon_1\varepsilon_2+D\varepsilon_1\varepsilon_2^2|$$

$$\le (1+\tfrac{1}{2}+|D|+\tfrac{1}{2}+|D|)\,\varepsilon_m.$$

Therefore, this algorithm is backstable.

b.) Data $x \in \mathbb{R}$, $f(x) = 2x$.

Solution:

$$\tilde{f}(x) = fl(x) \oplus fl(x)$$
$$= x(1+\varepsilon_1) \oplus x(1+\varepsilon_1)$$
$$= 2x(1+\varepsilon_1)\cdot(1+\varepsilon_2)$$
$$= f(x(1+\varepsilon_1)(1+\varepsilon_2))$$
$$= f(\tilde{x})$$

Now,

$$\frac{|x-\tilde{x}|}{|x|} = \frac{|x\varepsilon_1 + x\varepsilon_2 + x\varepsilon_1\varepsilon_2|}{|x|} \leq 3\varepsilon_m.$$

C.) Data $x \in \mathbb{R}$, $f(x) = x/x = 1$.

Solution:

$$\tilde{f}(x) = fl(x) \oslash fl(x)$$
$$= \frac{x(1+\varepsilon_1)}{x(1+\varepsilon_1)} \cdot (1+\varepsilon_2)$$
$$= 1+\varepsilon_2$$
$$\neq f(\tilde{x}).$$

Consequently, $\tilde{f}$ cannot be backwards stable. However,

$$\frac{|\tilde{f}(x) - f(x)|}{|f|} = \varepsilon_2$$

which implies the algorithm is stable.

d.) Data $x \in \mathbb{R}$, $f(x) = x - x$.

Solution:

$$\tilde{f}(x) = fl(x) \ominus fl(x)$$
$$= x(1+\varepsilon_1) \ominus x(1+\varepsilon_1)$$
$$= [x(1+\varepsilon_1) - x(1+\varepsilon_1)](1+\varepsilon_2)$$
$$= 0.$$

This algorithm is backstable.

$\blacksquare$

#15.2

a.) For this algorithm to be backwards stable the matrices $\tilde{U}$ and $\tilde{V}$ would have to be unitary.

b.) Clearly, this algorithm cannot be backwards stable do to roundoff error in $\tilde{U}, \tilde{V}$. That is, the columns of $\tilde{U}, \tilde{V}$ will not be orthonormal.

c.) Stability implies that $\tilde{U}, \tilde{V}, \tilde{\Sigma}$ are relatively close to U, V, Σ. I.e.

$$\frac{|\tilde{U}-U|}{|U|} = \mathcal{O}(\varepsilon_m), \quad \frac{|\tilde{V}-V|}{|V|} = \mathcal{O}(\varepsilon_m), \quad \frac{|\tilde{\Sigma}-\Sigma|}{|\Sigma|} = \mathcal{O}(\varepsilon_m).$$

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