Homework #9

#1,

Suppose $a,b \in \mathbb{R}$. Show that $\Lambda = \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}$ is similar to $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = M$ by diagonalizing M. How would you geometrically describe the linear transformation given by M?

Solution:

The eigenvalues of M satisfy:

$$\det(\lambda I - M) = 0$$
$$\Rightarrow (\lambda - a)^2 + b^2 = 0$$
$$\Rightarrow \lambda = a \pm ib$$

Since the eigenvalues obviously have algebraic multiplicity 1 it follows that $\Lambda \sim M$.

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#2.

Prove that eigenvalues of a projector can have no other value than zero or one.

Solution:

Let $\vec{v}$ be an eigenvector of P with eigenvalue λ. It follows that

$$\lambda^2 \vec{v} = P^2 \vec{v} = P\vec{v} = \lambda \vec{v}$$
$$\Rightarrow \lambda = 0, 1.$$

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#24.1

a.) True
b.) False
c.) True
d.) True
e.) False
f.) True
g.) True

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#24.4

a.) Let $A \in \mathbb{C}^{n\times n}$ satisfy $\rho(A) < 1$. Let $T \in \mathbb{C}^{n\times n}$, $Q \in \mathbb{C}^{n\times n}$ satisfy T is upper triangular, Q is unitary and

$$A = QTQ^*.$$

Define a sequence $\varepsilon_i \in \mathbb{C}$ satisfying

1. For all $i \neq j$, $T_{ii} + \varepsilon_i \neq T_{jj} + \varepsilon_j$
2. $\max_{1\le i\le n} |T_{ii}| + |\varepsilon_i| < 1$.

and let

$$T' = T + \begin{bmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{bmatrix}.$$

Therefore, since $T - T'$ is diagonal it follows that

$$\begin{aligned} A &= QTQ^* \\ &= Q(T + T' - T')Q^* \\ &= T - T' + QT'Q^*. \end{aligned}$$

By construction, T' is diagonalizable with $\rho(T') < 1$. Therefore,

$$\begin{aligned} \|A^n\| &= \left\| \sum_{r=0}^{n} \binom{n}{r} (T - T')^{n-r} \cdot QXD^rX^{-1}Q^* \right\|, \\ &\le \sum_{r=0}^{n} \binom{n}{r} \|T - T'\|^{n-r} \|X\| \cdot \|X^{-1}\| \cdot \|D\|^r \\ &\le \sum_{r=0}^{n} \binom{n}{r} \max_{1\le i\le n} |\varepsilon_i|^{n-r} K(X) \cdot \max_{1\le i\le n} |D_{ii}|^r \\ &\le \left(\max_{1\le i\le n} |\varepsilon_i| + \rho(T') \right)^n K(X). \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|A^n\| = 0.$$

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b.) Let $A \in \mathbb{C}^{n\times n}$ satisfy $\alpha(A)<0$. By the Schur factorization theorem there exists $T\in\mathbb{C}^{n\times n}$, $Q\in\mathbb{C}^{n\times n}$ such that

$$A = QTQ^*,$$

where T is upper triangular and Q is unitary. Define a sequence $\varepsilon_j \in \mathbb{C}$ satisfying

1. For all $i\neq j$, $T_{ii}+\varepsilon_i \neq T_{jj}+\varepsilon_j$,
2. $Re(T_{ii}+\varepsilon_i)<0$, $Re(\varepsilon_i)<0$

Therefore, letting $T' = T + \begin{bmatrix}\varepsilon_1 & 0\\ 0 & \varepsilon_n\end{bmatrix}$ it follows that

$$\|\exp(tA)\| = \|\exp(tQ(T+T'-T')Q^*)\|$$

$$= \|I + tQ(T+T'-T')Q^* + \tfrac{1}{2} t^2[Q(T+T'-T')Q^*]^2+\dots\|$$

$$\le \|\exp(t(T-T'))\|\cdot\|\exp(tT')\|.$$

By construction $T-T'$ is diagonal and T' is diagonalizable. Consequently,

$$\lim_{t\to\infty}\|\exp(tA)\| \le \max_{1\le i\le n} e^{-t|\varepsilon_i|}\max_{1\le j\le n} e^{-t|T_{ii}+\varepsilon_i|} = 0.$$

MST Problem

Define A by $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x+y \end{bmatrix}$.

- $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A^2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = A\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_1+F_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}$

  Assume for induction that

$$A^n\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$

$$\Rightarrow A^{n+1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = A\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_{n+1}+F_n \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix}.$$

- A has the matrix representation:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

  The eigenvectors and eigenvalues are:

$$\lambda_1 = \frac{1}{2}(1+\sqrt{5}),\quad \vec{V}_1 = \begin{bmatrix} \frac{1}{2}(-1+\sqrt{5}) \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2}(1-\sqrt{5}),\quad \vec{V}_2 = \begin{bmatrix} \frac{1}{2}(-1-\sqrt{5}) \\ 1 \end{bmatrix}$$

- In the eigen basis

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{(-1-\sqrt{5})\vec{V}_1 - (-1+\sqrt{5})\vec{V}_2}{-2\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})\vec{V}_1 - (1-\sqrt{5})V_2}{2\sqrt{5}}.$$

$$\Rightarrow A^n\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} \cdot \vec{V}_1 - \frac{1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\vec{V}_2$$

$$\Rightarrow F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$