Lecture 24: Eigenvalues

Definitions: Let $A \in \mathbb{R}^{n \times n}$, $\vec{v} \in \mathbb{C}^n$ is an eigenvector of with corresponding eigenvalue $\lambda \in \mathbb{C}$ if

$$A\vec{x} = \lambda \vec{x}.$$

example:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} \in \text{Null}(\lambda I - A)$$

$$\Rightarrow \det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$

$$\Rightarrow \begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad (\text{Find } [x, y]^T \text{ in Null}(A))$$

$$\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix} {}_{+iR1} \longrightarrow \begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow ix - y = 0$$

$$x = -iy$$

Eigenvectors:

$$\vec{x}_1, \vec{x}_2 = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow A\underbrace{\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}}_{X}\underbrace{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{\Lambda}$$

$$\Rightarrow A = X \Lambda X^{-1} \rightarrow \text{diagonalization!!!}$$

what e^{tA}??

$$e^x = 1 + x + x^2/2 + x^3/6 + \dots$$

$$e^{tA} = I + tA + \frac{t^2 A^2}{2} + \dots$$

$$\Rightarrow e^{tA} = I + t X \Delta X^{-1} + \frac{t^2}{2} X \Delta X^{-1} X \Delta X^{-1} + \dots$$

$$= I + t X \Delta X^{-1} + \frac{t^2}{2} X \Delta^2 X^{-1} + \dots$$

$$= X\left(I + t\Delta + \frac{t^2}{2}\Delta^2 + \dots \right)$$

$$= X\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} -i^2 & 0 \\ 0 & i^2 \end{bmatrix} + \dots \right) X^{-1}$$

$$= X\left(\begin{bmatrix} 1 - it + (-it)^2/2 + \dots & 0 \\ 0 & 1 + it + (it)^2/2 + \dots \end{bmatrix} \right) X$$

$$= X\left(\begin{bmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{bmatrix} \right) X^{-1}$$

$$= X\left(\begin{bmatrix} \cos(t) - i\sin(t) & 0 \\ 0 & \cos(t) + i\sin(t) \end{bmatrix} \right) X^{-1}$$

example:

If $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, what is $\sqrt{A}$??

$$A = I + \underbrace{\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}_{B} \qquad (I+B)^{1/2} = I + \tfrac{1}{2}B - \tfrac{1}{8}B^2 + \dots$$

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \dots$$

$$B^2 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(I+B)^{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \sqrt{A}.$$

Definitions-

1. λ is an eigenvalue. The eigenspace E_λ is the span of the eigenvectors corresponding to λ.
2. $\dim(E_\lambda)$ is the geometric multiplicity of λ
3. The characteristic polynomial of $A \in \mathbb{R}^{n \times n}$ is $p_A(z) = \det(zI - A)$.
4. The algebraic multiplicity of λ is its multiplicity as a root

Similarity Transformations

If $A \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{n \times n}$ is nonsingular then the map
$$A \mapsto X^{-1} A X$$
is called a similarity transformation. We say A, X are similar if there exists a similarity transformation between them.

Theorem - If X is nonsingular, then A and $X^{-1}AX$ have the same characteristic polynomial, eigenvalues, and multiplicities.

proof:

1. $$\begin{aligned} p_{X^{-1}AX}(z) &= \det(zI - X^{-1}AX) \\ &= \det(X^{-1}(zI - A)X) \\ &= \det(zI - A) \end{aligned}$$

 This proves everything except geometric multiplicities.

2. Let E_λ be an eigenspace of A. Let $\tilde{E}_\lambda = X^{-1}E_\lambda$. Then,

$$\begin{aligned} X^{-1}AX(X^{-1}E_\lambda) &= X^{-1}AE_\lambda \\ &= X^{-1}E_\lambda. \end{aligned}$$

Theorem - The algebraic multiplicity is at least as great as the geometric multiplicity.

proof.

Let λ be an eigenvalue with geometric multiplicity m_g. Form the matrix $\hat{V}$ whose m_g columns are an orthonormal basis for E_λ. Extend $\hat{V}$ to a unitary matrix V

$$\Rightarrow B = V^*AV = \begin{array}{c} \scriptstyle n\times m_g \\ \begin{bmatrix} \lambda I & C \\ 0 & D \end{bmatrix} \end{array}$$

$$\Rightarrow \det(zI - B) = \det(zI - \lambda I)\cdot\det(zI - D).$$

$$\Rightarrow m_a \geq m_g.$$

Definition - A matrix is defective if $m_a > m_g$.

Theorem - A matrix is nondefective if and only if it has an eigenvalue decomposition

$$A = X \Lambda X^{-1}.$$

Hermitian Matrices.

$A \in \mathbb{R}^{n \times n}$ is Hermitian if $A^* = A$.

Theorem - The eigenvalues of a are real with corresponding orthogonal eigenvectors.

Schur Factorization.

A Schur factorization of a matrix A is

$$A = QTQ^*,$$

$T \sim$ upper triangular

$Q \sim$ unitary.

Theorem - Every square matrix $A \in \mathbb{R}^{n \times n}$ has a Schur factorization

proof:

We prove this by induction on n.

1. $n=1$ is trivial.
2. Let $n \geq 2$. Let $\vec{v}$ be any eigenvector with eigenvalue λ. Assume $\|\vec{v}\| = 1$ and let is be the first column of a unitary matrix U.

$$\Rightarrow U^* A U = \begin{bmatrix} \lambda & B \\ 0 & C \end{bmatrix}$$

By inductive hypothesis there exists a Schur factorization of C:

$$VTV^* = C.$$

Let

$$Q = U \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}$$

$$\Rightarrow Q^* A Q = \begin{bmatrix} 1 & 0 \\ 0 & V^* \end{bmatrix} U^* A \cdot U \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & V^* \end{bmatrix} \begin{bmatrix} \lambda & B \\ 0 & C \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & B + V^* C \\ 0 & V^* C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} \lambda & B + T \\ 0 & T \end{bmatrix}.$$