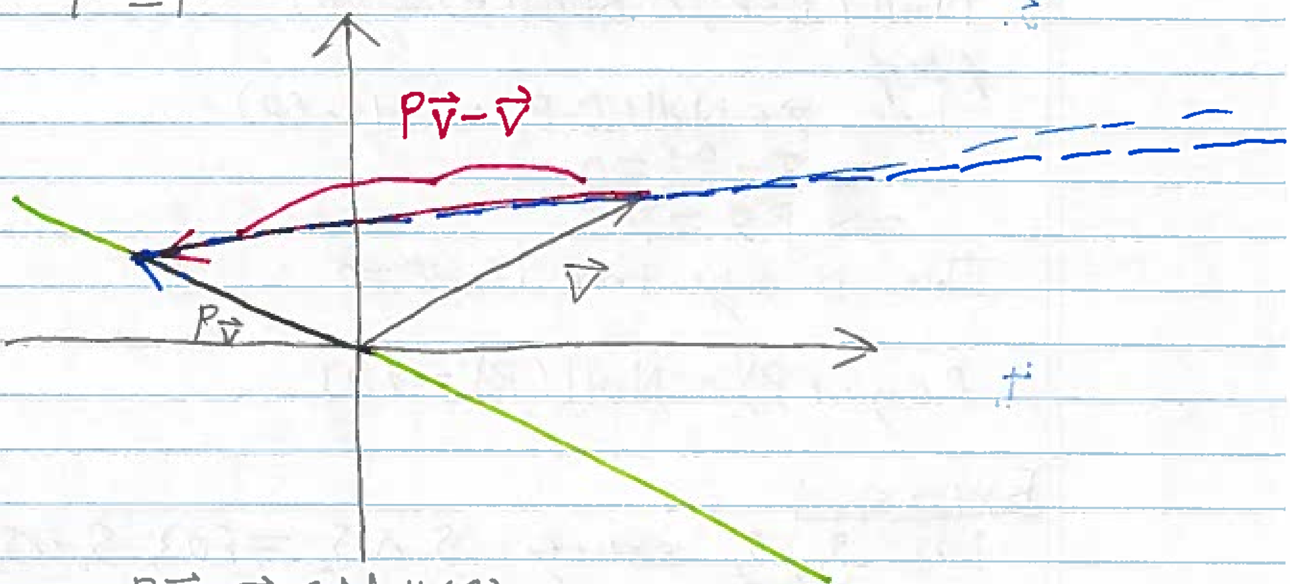
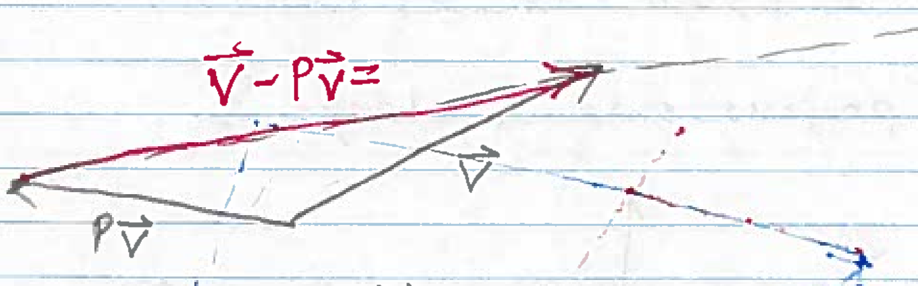
Lecture 6: Projections / QR algorithm

A projector $P \in \mathbb{R}^{n\times n}$ satisfies
$P^2 = P$



$\Rightarrow P\vec{v}-\vec{v} \in Null(P)$.
$P(P\vec{v}-\vec{v}) = P\vec{v} - P\vec{v} = 0$

P is a projection so is $I-P$



1. $Range(I-P) = Null(P)$

proof

1. Let $\vec{x} \in Range(I-P)$. There exists $\vec{v}$ such that
$\vec{x} = \vec{v} - P\vec{v}$
$\Rightarrow P\vec{x} = P\vec{v} - P\vec{v} = 0$.

2. Let $\vec{x} \in Null(P)$. Then,
$P\vec{x} = 0$
$\Rightarrow \vec{x} - P\vec{x} = \vec{x}$
$\Rightarrow (I-P)\vec{x} = \vec{x}$.
$\Rightarrow \vec{x} \in Range(I-P)$.

2. $\text{Null}(I-P) = \text{Range}(I-(I-P)) = \text{Range}(P)$

3. $\text{Null}(I-P) \cap \text{Null}(P) = \{0\}$

~~proof~~:

Let $\vec{x} \in \text{Null}(I-P) \cap \text{Null}(P)$

$\Rightarrow \vec{x} - P\vec{x} = 0$

$\Rightarrow P\vec{x} = \vec{x}$

This is only true if $\vec{x} = 0$.

4. $\text{Range}(P) \cap \text{Null}(P) = \{0\}$

Extension:

Let $\underbrace{S_1, S_2}_{\text{subspaces!}}$ satisfy $S_1 \cap S_2 = \{0\}$, $S_1 \cup S_2 = \mathbb{R}^n$

Define P by

1. $P\vec{x} = \vec{x}$ if $\vec{x} \in \text{span}\{S_1\}$
2. $P\vec{x} = 0$ if $\vec{x} \in \text{span}\{S_2\}$

P projects onto S_1 along S_2.

What is the rank of projection onto span of a vector?

$\to$ Rank 1!

Orthogonal Projection:

P is an orthogonal projection if $S_1 \perp S_2$.

Theorem - A projection is orthogonal if and only if $P^T = P$

proof

1. Suppose $P \in \mathbb{R}^{m \times m}$ projects onto S_1 along S_2 with $S_1 \perp S_2$. Let $\text{span}\{q_1, \dots, q_n\} = S_1$, $\text{span}\{q_{n+1}, \dots, q_m\} = S_2$ with $\{q_1, \dots, q_m\}$ an orthonormal basis.

$$\Rightarrow P\vec{q}_j = \begin{cases} \vec{q}_j & \text{if } 1 \le j \le n \\ 0 & \text{if } n+1 \le j \le 1. \end{cases}$$

$$\Rightarrow PQ = \left[\vec{q}_1 \middle| \cdots \middle| \vec{q}_n \middle| 0 \middle| \cdots \middle| 0 \right]$$

$$\Rightarrow Q^T P Q = \begin{bmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & 1 & & \\ & 0 & & 0 & \\ & & & & \ddots \; 0 \end{bmatrix}$$

$$\Rightarrow P = Q \Sigma Q^*.$$

2. Suppose $P^T = P$. Let $x \in S_1$, $y \in S_2$. Then,

$$\begin{aligned} \vec{x}^T \vec{y} = (P\vec{x})^T (I - P)\vec{y}^T &= \vec{x}^T P^T (I - P\vec{y}^T) \\ &= \vec{x}^T (P\vec{y} - P^2 \vec{y}) \\ &= 0. \end{aligned}$$

Comment:

The reduced SVD of P is simply

$$P = \hat{Q}\hat{Q}^T$$

Projection Orthonormal Basis:

$$\vec{y} = \underbrace{\hat{Q}}_{m \times n} \; \underbrace{\hat{Q}^T}_{n \times m} \vec{v}$$

$\hat{Q}\hat{Q}^*$ projects onto column space of $\hat{Q}$.

Any projector can be constructed from orthonormal matrices.

Rank-one orthogonal projectors:

$$P_q = q q^T.$$

example:

What is the matrix that projects onto the

$$\text{range}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \right\}.$$

Idea: Find orthonormal basis $\vec{q}_1, \vec{q}_2$

$$\text{span}\{\vec{q}_1, \vec{q}_2\} = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \right\}.$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 \end{array}$$

$$\vec{q}_1 = r_{11} \vec{a}_1$$

$$\|\vec{q}_1\| = 1 \Rightarrow r_{11} \cdot \sqrt{3}$$

$$\Rightarrow r_{11} = 1/\sqrt{3}.$$

$$\vec{a}_2 = r_{12}\,\vec{q}_1 + r_{22}\,\vec{q}_2$$

$$\Rightarrow \vec{q}_1^{\,T} \cdot \vec{a}_2 = r_{12}$$

$$\Rightarrow r_{12} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}$$

$$\frac{\vec{a}_2 - r_{12}\vec{q}_1}{r_{22}} = \vec{q}_2$$

$$\|\vec{q}_2\| = 1 \Rightarrow \left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{3}} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right)^2 = r_{22}^2$$

$$\Rightarrow \quad r_{22}^2 = 4/9 + \frac{16}{9} + 4/9$$

$$r_{22}^2 = \frac{24}{9}$$

$$r_{22} = \frac{2\sqrt{6}}{3}$$

$$\Rightarrow \vec{q}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{q}_2 = \frac{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - 1/\sqrt{3} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}}{\frac{2 \cdot \sqrt{6}}{3}}$$

$$\Rightarrow \vec{q}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\vec{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Bonus:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \underset{Q}{\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}} \underset{R}{\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2\sqrt{6}/3 \end{bmatrix}}$$

$\uparrow$

This is the QR factorization!!

QR-Algorithm

$$\underset{A}{\left[\vec{a}_1 \middle| \cdots \middle| \vec{a}_n\right]} = \underset{Q}{\left[\vec{q}_1 \middle| \cdots \middle| \vec{q}_n\right]} \underset{R}{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}}$$

$$\vec{a}_1 = r_{11}\vec{q}_1$$
$$\vec{a}_2 = r_{12}\vec{q}_1 + r_{22}\vec{q}_2$$
$$\vdots$$
$$\vec{a}_n = r_{1n}\vec{q}_1 + r_{2n}\vec{q}_2 + \dots + r_{nn}\vec{q}_n$$

Cond $\|\vec{q}_i\| = 1$

1. $\vec{q}_1 = \dfrac{\vec{a}_1}{r_{11}}$, $\vec{q}_1^T\vec{a}_1 = r_{11}$

2. $\vec{q}_2 = \dfrac{\vec{a}_2 - r_{12}\vec{q}_1}{r_{22}}$, $\vec{q}_1^T\vec{a}_2 = r_{12}$, $\vec{q}_2\vec{a}_2 = r_{22}$

$\vdots$

j. $\vec{q}_j = \dfrac{\vec{a}_j - r_{1j}\vec{q}_1 - r_{2j}\vec{q}_2 - \dots - r_{j-1,j}\vec{q}_{j-1}}{r_{jj}}$, $\vec{q}_i\vec{a}_j^T = r_{ij}$

Arbitrary Projection

Construction of orthogonal projection onto range(A), $A \in \mathbb{R}^{m \times n}$

$$A = [\vec{a}_1 | \cdots | \vec{a}_n]$$

$$\Rightarrow \text{range}(A) = \text{span}\{\vec{a}_1, \dots, \vec{a}_n\}.$$

Let P be the projection, let $\vec{v} \in \mathbb{R}^m$.

$$\Rightarrow P\vec{v} = \vec{y} \in \mathbb{R}^m$$

$$\Rightarrow P\vec{v} - \vec{v} \in \text{Null}(P) = \text{Range}(P)^{\perp}.$$

Therefore, for all j,

$$\vec{a}_j^*(P\vec{v} - \vec{v}) = 0$$

Since $P\vec{v} \in \text{range}(A)$, $\exists \vec{x}$ such that $A\vec{x} = P\vec{v}$.

$$\Rightarrow \vec{a}_j^*(A\vec{x} - \vec{v}) = 0$$

$$\Rightarrow A^*(A\vec{x} - \vec{v}) = 0$$

$$\Rightarrow A^*A\vec{x} = A^*\vec{v}$$

$$\Rightarrow \vec{x} = (A^*A)^{-1}A^*\vec{v}$$

$$\Rightarrow A\vec{x} = A(A^*A)^{-1}A^*\vec{v} = P\vec{v}$$

$$\Rightarrow P = A(A^*A^{-1})A^*\vec{v}.$$