Lecture 12: Conditioning

$\mathbb{X}, \mathbb{Y}, \rightarrow$ normed vector spaces
$(\mathbb{X} \subset \mathbb{R}^n, \|\cdot\|_p)$
$(\mathbb{Y} \subset \mathbb{R}^m \ \|\cdot\|_q)$

A proplem is a function $f: \mathbb{X} \mapsto \mathbb{Y}$ from data to solutions.

examples

1. $A\vec{x} = \vec{b}$, $f = A$
2. $\frac{dx}{dt} = F(x)$, f maps intial condition to solution curves.
   $x(0) = x_0$
3. $p(x) = a_0 + a_1 x + a_2 x^2$, f maps $\mathbb{R}^3 \mapsto \mathbb{R}^3$ (coefficients into roots)

* A problem is well-conditioned if small changes in x lead to small changes in $f(x)$

* A problem is ill-conditioned if there exists an x such that small changes in x lead to large change in $f(x)$.

Absolute condition number:

$$\hat{K}(\vec{x}) = \lim_{\delta \to 0} \max_{\|\delta x\| \le \delta} \frac{\|\delta f\|}{\|\delta x\|} = \max_{\|\delta x\| \le \delta} \frac{\|f(\vec{x}+\delta x) - f(\vec{x})\|}{\|\delta x\|}$$

If f is differentiable recall that:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \rightarrow \text{Jacobian matrix} \quad \hat{K}(\vec{x}) = \|J(x)\|$$

example:

1. $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x/2$
$f'(x) = 1/2$
$\Rightarrow \hat{K} = 1/2$ (well-conditioned)

2. $f: \mathbb{R} \to \mathbb{R}$
$f(x) = \sqrt{x}$
$f'(x) = \frac{1}{2} x^{-1/2}$

$\Rightarrow \hat{K} = \frac{1}{2x^{-1/2}}$ (blows up when $x \to 0$).

3. $f: \mathbb{R}^2 \to \mathbb{R}$
$f(x) = x_2 - x_1$
$J = [-1, \ 1]$

$\hat{K} = \|J\|_\infty = 2$

Relative Condition Number

$$K = \lim_{\delta \to 0} \max_{\|\delta x\| \le \delta} \left(\frac{\|\delta f\|}{\|f(x)\|} \Big/ \frac{\|\delta x\|}{\|x\|} \right)$$

$$\Rightarrow K = \frac{\|J(x)\|}{\|f(x)\| / \|x\|}$$

* A problem is well (ill) conditioned if K is small (large).

examples:

1. $f(x) = x/2$
$K = 1$

2. $f(x) = \sqrt{x}$
$\frac{1}{2x^{-1/2}} \Big/ \sqrt{x}/x = 1/2$

3. $K = \frac{2}{|x_1 - x_2| \max\{|x_1|, |x_2|\}}$
$\downarrow$
ill conditioned if $x_1 \simeq x_2$!

example:

eigenvalues of

$$\begin{bmatrix} 1 & 1000 \\ 0 & 1 \end{bmatrix} \to \lambda = 1, -1 \qquad \|f\|_\infty = 1$$

$$f : \mathbb{R}^{n \times n} \longmapsto \mathbb{R}$$

$$\begin{bmatrix} 1 & 1000 \\ -.001 & 1 \end{bmatrix} \to \lambda = 0, 2$$

$$\boxed{\delta f = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \|\delta f\|_\infty = 3.}$$

Input was a matrix:

$$\|\delta x\|_\infty = 10^{-3}$$

$$\|x\| = 10^{3}$$

$$\Rightarrow \hat{K} > \left(\frac{\|\delta f\|}{\|f\|} \Big/ \frac{\|\delta x\|}{\|x\|} \right)$$

$$> 3 / 10^{-6}$$

$$= 3 \cdot 10^{6} \to 3 \text{ million!!!} \quad \text{(very ill-conditioned)}$$

Matrix Vector Multiplication

$Ax \qquad f : \mathbb{R}^n \to \mathbb{R}^n$

$$\delta f = A(x + \delta x) - Ax$$

$$= A\delta x$$

$$\Rightarrow K = \sup_{\delta x} \left(\frac{\|A\delta x\|}{\|Ax\|} \Big/ \frac{\|\delta x\|}{\|x\|} \right)$$

$$= \sup_{\delta x} \left(\frac{\|A\delta x\|}{\|\delta x\|} \Big/ \frac{\|x\|}{\|Ax\|} \right)$$

$$= \|A\| \cdot \frac{\|x\|}{\|Ax\|}$$

If A is square then

$$\frac{\|x\|}{\|Ax\|} \leq \|A^{-1}\|$$

$$\|A^{-1}\| = \sup_{y} \frac{\|A^{-1}y\|}{\|y\|},$$

let $y = Ax$

$$\Rightarrow \|A^{-1}\| \geq \frac{\|x\|}{\|Ax\|}$$

$$\Rightarrow K \leq \|A\| \cdot \|A^{-1}\|$$

In fact can pick x to make equal!

$$\boxed{K = \|A\| \cdot \|A^{-1}\| \quad \text{(condition number of a matrix)}}$$

In the two norm what is the condition number?

$$\|A\|_2 \cdot \|A^{-1}\| = \frac{\sigma_1}{\sigma_m} \rightarrow \text{ratio of singular values!!}$$

Perturbing A:

$$A\hat{x} = b$$

$$(A + \delta A)(x + \delta x) = b$$

$$\Rightarrow \delta A x + A\delta x + \cancel{\delta A \delta x} = 0$$

$$\delta A x + A\delta x = 0$$

$$\delta x = -A^{-1}(\delta A)x$$

$$\|\delta x\| \leq \|A^{-1}\| \cdot \|\delta A\| \cdot \|x\|$$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \Big/ \frac{\|\delta A\|}{\|A\|} \leq \|A^{-1}\| \cdot \|A\| = K(A)!$$