Lecture 14: Stability

Big $O$

$f = O(\phi(x))$ as $x \to x_0$ means there exists constants $K_0$ and $C$ so that

$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| \leq K_0 \phi(x_0)$

Examples:

1. $\sin(x)$

   $\rightarrow \sin(x) = O(1) \Rightarrow \sin(x) = O(\delta)$

   $\sin(x) \neq O(\delta)$

   Proof:
   
   $\sin(x) = k_0 x$

   $\Rightarrow \frac{\sin(x)}{x} = k_0$

   $\Rightarrow \lim_{x \to 0} \frac{\sin(x)}{x} = 1 = 0$

Algorithm:

problem: $y = \frac{x}{x}$

definition: $y = \frac{x}{x}$

Example:

$\tilde{f}: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$

$\tilde{f}: \mathbb{R} \to \mathbb{R}$ defined by $\tilde{f}(x) = \|f(x) - f(x_0)\|$

Accuracy:

Absolute error $AE = \|f(x) - f(x_0)\|$

Relative error $RE = \|f(x) - f(x_0)\| \|f(x)\|$
An algorithm is accurate if
\[ \frac{\|f(x) - s(x)\|}{\|s(x)\|} = O(\epsilon_{\text{machine}}). \]

\[ \|f(x) - f(x)\| = O(\epsilon_{\text{machine}}) \]

\[ \text{for some } x \text{ with } \frac{\|s(x) - x\|}{\|x\|} = O(\epsilon_{\text{machine}}). \]

\( \star \) An algorithm is stable if it gives nearly the right answer to nearly the right question.

\[ \frac{\|s(x) - f(x)\|}{\|f(x)\|} = O(\epsilon_{\text{machine}}) \]

\[ \text{for some } x \text{ with } \frac{\|f(x) - x\|}{\|x\|} = O(\epsilon_{\text{machine}}). \]

\( \star \) An algorithm is backward stable if it gives exactly the right answer to nearly the right question.

Examples:
\[ f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x_1, x_2) = x_1 - x_2 \]
\[ f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x_1, x_2) = f_1(x_2) \oplus f_1(x_1) \]

\[ f_1(x_1) = x_1(1 + \epsilon_1) \]
\[ f_1(x_2) = x_2(1 + \epsilon_2) \]

\[ f_1(x_2) \circ f_1(x_1) = (f_1(x_2) - f_1(x_1))(1 + \epsilon_3) \]
\[ = (x_2(1 + \epsilon_2) - x_1(1 + \epsilon_1))(1 + \epsilon_3) \]
\[ = x_2(1 + \epsilon_2 + \epsilon_3 + \epsilon_2 \epsilon_3) - x_1(1 + \epsilon_1 + \epsilon_3 + \epsilon_2 \epsilon_3) \]
\[ = x_2(1 + \epsilon_4) - x_1(1 + \epsilon_5) \]
\[ |e_1| \leq 2 \varepsilon_{\text{machine}} + O(\varepsilon_{\text{machine}}^2) \]
\[ |e_2| \leq 2 \varepsilon_{\text{machine}} + O(\varepsilon_{\text{machine}}^2) \]

\[
\Rightarrow \delta(x_1, x_2) - \delta(x_1(1 + e_1), x_2(1 + e_2)) = 0
\]

\[
\|x - x_1\| = \|((e_1 x_1, e_2 x_2))\| \leq \max \{\varepsilon_1, \varepsilon_2\} \|x\| \leq \varepsilon_{\text{machine}}.
\]

**Accuracy of Backward Stable:***

**Theorem:** If a backward stable algorithm is applied to solve a problem \( f: \mathbb{R} \rightarrow \mathbb{R} \) with condition number \( N \), then

\[
\|f(x) - \delta(x)\| = O(N(x) \varepsilon_{\text{machine}})
\]

**Proof:**

By backward stability:

\[ \delta(x) = f(x) \]

For some \( x \) satisfying

\[ \|x - x_1\| = O(N(x) \varepsilon_{\text{machine}}) \]

From the definition of condition number

\[
K(x) \geq \frac{\|f(x + e) - f(x)\|}{\|e\|} \frac{\|f(x)\|}{\|e\|}
\]

\[
= \frac{\|f(x) - f(x)\|}{\|f(x)\|} \frac{\|x - x_1\|}{\|x - x_1\|}
\]

\[
\Rightarrow \|x - x_1\| \geq \frac{\|f(x) - f(x)\|}{\|f(x)\|}
\]

\[
\Rightarrow \|f(x) - f(x)\| \leq \varepsilon_{\text{machine}} \|x\| \Rightarrow \|f(x) - f(x)\| = O(N(x) \varepsilon_{\text{machine}}) \]

\[
\Rightarrow \|f(x) - f(x)\| \leq \varepsilon_{\text{machine}} \|x\|
\]
Backward stability + conditioning number

\[ \Rightarrow \text{accuracy} \]

**Examples:**

1. \( f(x_1, x_2) = x_1 \cdot x_2 \)
   
   \[ f'(x_1, x_2) = f'(x_1) \otimes f'(x_2) \]
   
   \[ = x_1(1+\epsilon_1) \otimes x_2(1+\epsilon_2) \]
   
   \[ = [x_1(1+\epsilon_1), x_2(1+\epsilon_2)](1+\epsilon) \]
   
   \[ = x_1(1+\epsilon_1+\epsilon_2+\epsilon_2\epsilon) \cdot x_2(1+\epsilon_2) \]
   
   \[ = f(x_1(1+\epsilon_1+\epsilon_2+\epsilon_2\epsilon), x_2(1+\epsilon_2)) \]

2. \( f(x_1, x_2) = x_1 x_2 = x_1 y_1 + x_2 y_2 \)
   
   \[ f'(x_1, x_2) = [f'(x_1) \otimes f'(x_2)] \]
   
   \[ = (x_1 y_1) \oplus (x_2 y_2) \]

\[ \|x_1 - x_1\| = \|x_1 y_1 + x_2 y_2\| \]

\[ \|x_1\| \]

\[ \leq \max(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \|x_1 - x_1\| \]

\[ \|x_1\| \]

\[ = \Theta(\epsilon_{\text{machine}}) \]

\[ \epsilon_1^2 + \epsilon_2^2 \]

\[ \epsilon_1^2 + \epsilon_2^2 \]

\[ \Rightarrow f(x_1, x_2) = (x_1 y_1 + \epsilon_2 y_2)(1+\epsilon) \]

\[ = f(x_1, x_2) \]
$$\frac{x}{y} = \left( x_1 (1 + \varepsilon_1) \right)^{\frac{y}{x}} \cdot \left( x_2 (1 + \varepsilon_2) \right)^{\frac{y}{x}}$$

$$\frac{y}{x} = \left( y_1 (1 + \varepsilon_1) \right)^{\frac{x}{y}} \cdot \left( y_2 (1 + \varepsilon_2) \right)^{\frac{x}{y}}$$

$$\left( 1 + \varepsilon_1 \right)^{\frac{y}{x}} = 1 + \frac{1}{2} \varepsilon_1 + O(\varepsilon_1^2)$$

$$\frac{\| (x, y) - (x, y) \|}{\| (x, y) \|} = \frac{\| x_1 (1 + \varepsilon_1) - x_1 x_2 (1 + \varepsilon_2) - x_1 x_2 (1 + \varepsilon_3) - x_1 x_2 (1 + \varepsilon_4) \|}{\| (x_1, x_2, y_1, y_2) \|}$$

$$\leq O(\varepsilon_1)$$

**Example:**

$$x + 1 = f(x)$$

$$\frac{x}{x} = f(1(x) + 1).$$

$$= x (1 + \varepsilon_1) + 1$$

$$= x (1 + \varepsilon_1) (1 + \varepsilon_2)$$

$$= x \left( 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1 \varepsilon_2 \right) + \varepsilon_2 + 1$$

$$= f(x)$$

$$\| x - x \| = \| x (\varepsilon_1 + \varepsilon_2 + \varepsilon_1 \varepsilon_2) + \varepsilon_2 \|$$

$$\leq \| x \|$$

$$\Rightarrow \text{diverges if } x = 0.$$
Example:

Our product

\[ A = xy^* \]

Not possible to be backwards stable.

\[ xy^* = \text{rank } 1 \]

\[ \mathbf{F}(x, y) = \begin{bmatrix}
F_1(x_1) \otimes f_1(y_1) & \cdots & F_1(x_n) \otimes f_1(y_n) \\
\vdots & & \vdots \\
F_1(x_n) \otimes f_1(y_1) & \cdots & F_1(x_n) \otimes f_1(y_n)
\end{bmatrix} \rightarrow \text{Not rank } 1 \]