MST 351/651: Math Modeling

Lecture 1: Intro Example

Classic Physics: Derive a system of differential equations and solve them. The entire evolution of the system is determined from initial conditions.

Example: An object is placed in cold (hot) environment at a fixed temperature T_E

What is the temperature of the object as a function of time?

Mathematical model:

1. $$\frac{dT}{dt} = K(T_E - T), \quad T(0) = T_0$$

$\frac{dT}{dt}$: rate of change of temp. of object

K: proportionality constant

$(T_E - T)$: temperature difference.

$K \sim$ units of inverse time.

Newton's law of cooling: Rate of change of temperature is proportional to temp. difference.

Solution:

Let $\Delta t = T - T_E$ and $\Delta T_0 = T_0 - T_E$

$$\Rightarrow \frac{dT}{dt} = \frac{d\Delta t}{dt}$$

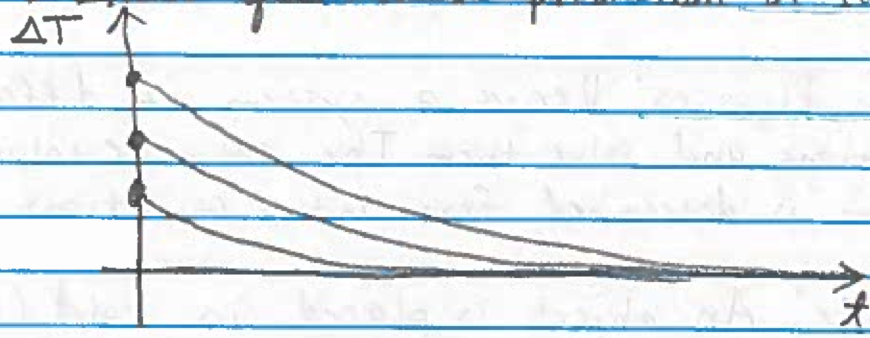
$$\Rightarrow \frac{d\Delta t}{dt} = -K\Delta T, \quad \Delta T(0) = \Delta T_0$$

$$\Rightarrow \int_{\Delta T_0}^{\Delta t} \frac{1}{\Delta T}\, d\Delta t = \int_0^t -K\, dt$$

$$\Rightarrow \ln(\Delta T) - \ln(\Delta T_0) = -Kt$$

$$\Rightarrow \Delta T = \Delta T_0 \exp(-Kt).$$

* Exact quantitative prediction of temperature



2. Now suppose the object and its environment influence each other.

$$\frac{dT}{dt} = -K_1(T - T_E), \quad T(0) = T_0$$

$$\frac{dT_E}{dt} = K_2(T - T_E) \quad T_E(0) = T_{E,0}$$

Let $\Delta T = T - T_E$, $H = T + T_E$.

$$\Rightarrow \frac{d\Delta T}{dt} = \frac{dT}{dt} - \frac{dT_E}{dt} = -(K_1 + K_2)\Delta T$$

$$\frac{dH}{dt} = \frac{dT}{dt} + \frac{dT_E}{dt} = -(K_1 - K_2)\Delta T$$

$$\Rightarrow \Delta T = \Delta T_0 \exp(-(K_1 + K_2)t)$$

$$\Rightarrow \frac{dH}{dt} = -(K_1 - K_2)\Delta T_0 \exp(-(K_1 + K_2)t)$$

$$\Rightarrow H(t) = \frac{(K_1 - K_2)}{(K_1 + K_2)}\Delta T_0 (e^{-(K_1+K_2)} - 1) + H_0$$

Observations:

1. $\Delta T \rightarrow 0$
2. $H \rightarrow \frac{K_2 - K_1}{K_1 + K_2}\Delta T_0 + H_0$

Example: Nonlinear Newton's Law of Cooling

$$\dot{\Delta T} = -K\Delta T - \alpha \Delta T^3$$

$$\Delta T(0) = \Delta T_0$$

<u>Note</u>: If we think of $\dot{\Delta T} = f(\Delta T)$ then this is the first two terms in a Taylor series expansion.

$$\Rightarrow -t = \int_{T_0}^{\Delta T} \frac{1}{Ks+\alpha s^3}\, ds$$

$$= \int_{T_0}^{\Delta T}\left(\frac{1}{Ks} + \frac{K-\alpha}{\alpha}\,\frac{1}{K+\alpha s^2}\right) ds$$

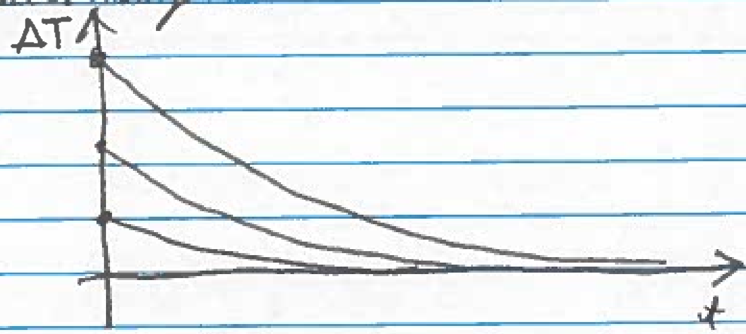
$$= \frac{1}{K}\ln\left(\frac{\Delta T}{\Delta T_0}\right) + \frac{K-\alpha}{K\alpha}\sqrt{\frac{K}{\alpha}}\left(\tan^{-1}\left(\sqrt{\frac{K}{\alpha}}\,\Delta T\right) - \tan^{-1}\left(\sqrt{\frac{K}{\alpha}}\,\Delta T_0\right)\right)$$

To get a complete solution we then need to invert.

Poincare took a different approach.

$$\dot{\Delta T} = 0 \quad \text{when} \quad \Delta T = 0$$

i.e. the object stops cooling when $\Delta T = 0$. Before that $\dot{\Delta T}$ is always negative implying ΔT is monotone decreasing.



All initial conditions go to 0!