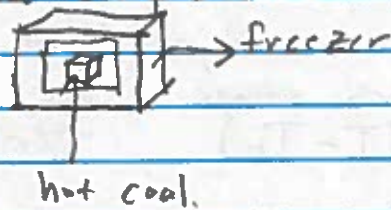


Lecture 4: Intro Example

Classic Physics: Derive a system of differential equations and solve them. The entire evolution of the system is determined from initial conditions.

Example: An object is placed in cold (hot) environment at a fixed temperature  $T_E$ .



What is the temperature of the object as a function of time?

Mathematical model:

1.  $\frac{dT}{dt} = K(T_E - T)$ ,  $T(0) = T_0$   
Annotations:  $\frac{dT}{dt}$  is the rate of change of temp. of object;  $K$  is the proportionality constant;  $K \sim$  units of inverse time;  $(T_E - T)$  is the temperature difference.

Newton's law of cooling: Rate of change of temperature is proportional to temp. difference.

Solution:

Let  $\Delta t = T - T_E$  and  $\Delta T_0 = T_0 - T_E$

$\Rightarrow \frac{dT}{dt} = \frac{d\Delta t}{dt}$

$\Rightarrow \frac{d\Delta t}{dt} = -K\Delta t$ ,  $\Delta T(0) = \Delta T_0$

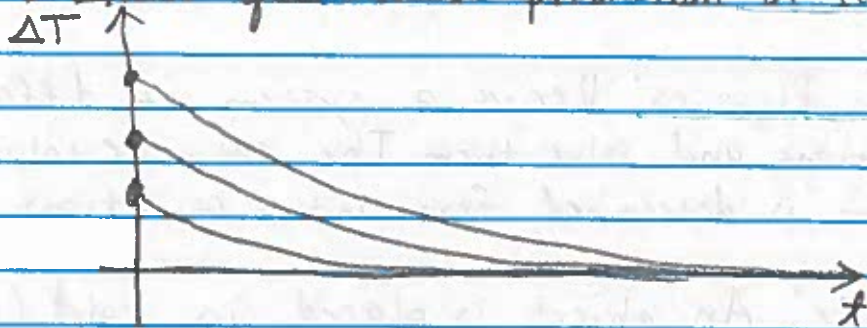
$\Rightarrow \int_{\Delta T_0}^{\Delta T} \frac{1}{\Delta T} d\Delta t = \int_0^t -K dt$

$\Rightarrow \ln(\Delta T) - \ln(\Delta T_0) = -Kt$

$\Rightarrow \Delta T = \Delta T_0 \exp(-Kt)$



\* Exact quantitative prediction of temperature



2. Now suppose the object and its environment influence each other.

$$\frac{dT}{dt} = -K_1(T - T_E), \quad T(0) = T_0$$

$$\frac{dT_E}{dt} = K_2(T - T_E), \quad T_E(0) = T_{E0}$$

$$\text{Let } \Delta T = T - T_E, \quad H = T + T_E$$

$$\Rightarrow \frac{d\Delta T}{dt} = \frac{dT}{dt} - \frac{dT_E}{dt} = -(K_1 + K_2)\Delta T$$

$$\Rightarrow \frac{dH}{dt} = \frac{dT}{dt} + \frac{dT_E}{dt} = -(K_1 - K_2)\Delta T$$

$$\Rightarrow \Delta T = \Delta T_0 \exp(-(K_1 + K_2)t)$$

$$\Rightarrow \frac{dH}{dt} = -(K_1 - K_2)\Delta T_0 \exp(-(K_1 + K_2)t)$$

$$\Rightarrow H(t) = \frac{(K_1 - K_2)}{(K_1 + K_2)} \Delta T_0 (e^{-(K_1 + K_2)t} - 1) + H_0$$

Observations:

1.  $\Delta T \rightarrow 0$

2.  $H \rightarrow \frac{K_1 - K_2}{K_1 + K_2} \Delta T_0 + H_0$

### Example: Nonlinear Newton's Law of Cooling

$$\dot{\Delta T} = -k\Delta T - \alpha\Delta T^3$$

$$\Delta T(0) = \Delta T_0$$

Note: If we think of  $\dot{\Delta T} = f(\Delta T)$  then this is the first two terms in a Taylor series expansion,

$$\Rightarrow -t = \int_{T_0}^{\Delta T} \frac{1}{kS + \alpha S^3} ds$$

$$= \int_{T_0}^{\Delta T} \left( \frac{1}{kS} + \frac{k-\alpha}{\alpha} \frac{1}{k + \alpha S^2} \right) ds$$

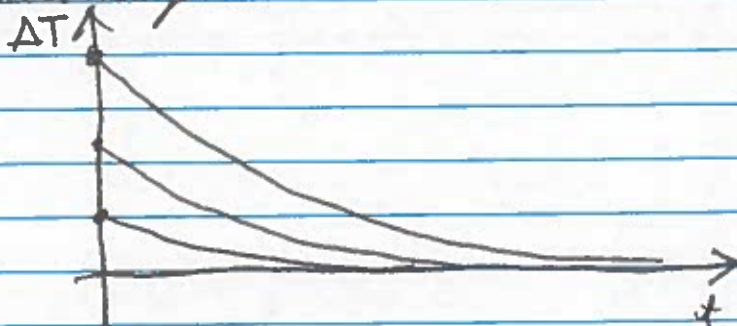
$$= \frac{1}{k} \ln \left( \frac{\Delta T}{\Delta T_0} \right) + \frac{k-\alpha}{k\alpha} \sqrt{\frac{k}{\alpha}} \left( \tan^{-1} \left( \sqrt{\frac{k}{\alpha}} \frac{\Delta T}{\sqrt{k}} \right) - \tan^{-1} \left( \sqrt{\frac{k}{\alpha}} \frac{\Delta T_0}{\sqrt{k}} \right) \right)$$

To get a complete solution we then need to invert.

Poincaré took a different approach.

$$\dot{\Delta T} = 0 \text{ when } \Delta T = 0$$

i.e. the object stops cooling when  $\Delta T = 0$ . Before that  $\dot{\Delta T}$  is always negative implying  $\Delta T$  is monotone decreasing.



All initial conditions go to 0!



Handwritten notes at the top of the page, including the equation  $T_A = T_A V = T_A$  and  $T_A = 2000$ .

Handwritten notes in the middle section, including the equation  $(T_A + T_A) = T_A$ .

Handwritten equation:  $sh = 1^{100} = 1 - 1 = 0$

Handwritten equation:  $(1 + 1) \cdot 100 = 200$

Handwritten equation:  $(1 + 1) \cdot 100 = 200$

Handwritten text: "The first part of the problem is to find the value of T\_A"

Handwritten text: "The second part is to find the value of T\_A"

Handwritten text: "The third part is to find the value of T\_A"

Handwritten text: "The fourth part is to find the value of T\_A"



Handwritten text: "The fifth part is to find the value of T\_A"