Lecture 10: Phase Portraits

Example:

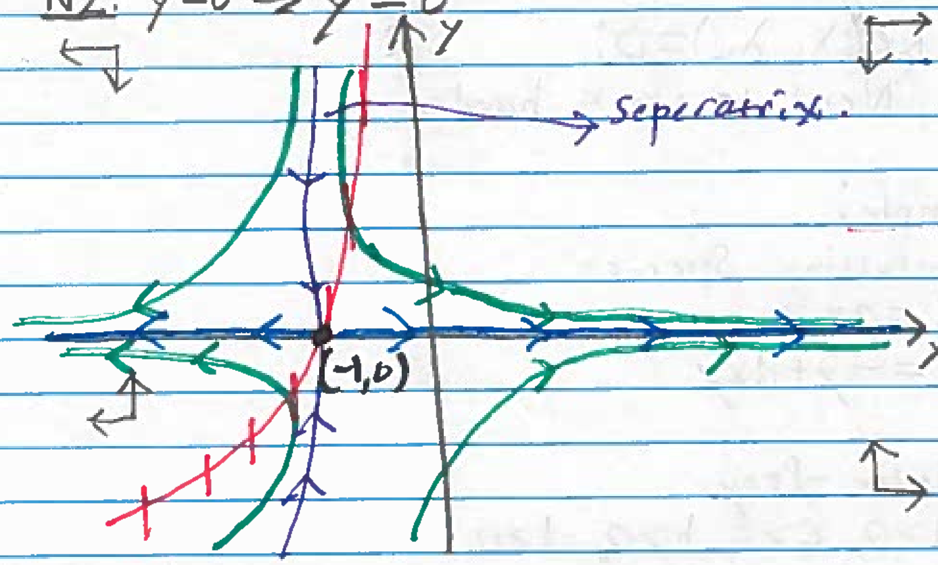
Sketch the phase portrait of

$$\begin{cases} \dot{x} = x + e^{-y} \\ \dot{y} = -y \end{cases}$$

Nullclines

N1: $\dot{x} = 0 \Rightarrow y = -\ln(-x)$

N2: $\dot{y} = 0 \Rightarrow y = 0$



Local Analysis:

Linearization about $(-1,0)$:

$$f(x,y) = f(x^*,y^*) + \nabla f\big|_{(x^*,y^*)} \cdot (\vec{x} - \vec{x}^*) + \dots$$

$$g(x,y) = g(x^*,y^*) + \nabla g\big|_{(x^*,y^*)} \cdot (\vec{y} - \vec{y}^*) + \dots$$

Near a fixed point

$$\dot{\vec{x}} = J(F)\big|_{(x^*,y^*)} (\vec{x} - \vec{x}^*)$$

Change Variables:

$$\tilde{x} = x - x^*, \quad y - y^*$$

$$\Rightarrow \dot{\tilde{\vec{x}}} = J(F)\big|_{(x^*,y^*)} \vec{x}^*$$

In this case $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow$ saddle fixed point.

Phase Plane Summary:

1. Draw nullclines
2. Draw direction arrows
3. Find equilibrium
4. Calculate Jacobian
5. Determine eigenvalues

$Re(\lambda_1, \lambda_2) \neq 0$:

hyperbolic fixed points

sign of $Re(\lambda_1, \lambda_2)$ determines stability.

$Re(\lambda_1, \lambda_2) = 0$:

Need to work harder

Example:

Interacting Species

$$\dot{x} = ax - bxy$$
$$\dot{y} = -cy + dxy$$

Predator-Prey

$a > 0,\ c > 0,\ b > 0,\ d > 0.$

Nullclines:

N1: $x = 0$ $(\dot{x} = 0)$

N2: $y = a/b$ $(\dot{x} = 0)$

N3: $y = 0$ $(\dot{y} = 0)$

N4: $x = c/d$ $(\dot{y} = 0)$

Fixed points: $(0,0)$, $(c/d, a/b)$

Local Analysis:

$$J = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \Rightarrow \lambda_{1,2} = a, -c \quad \text{(Saddle)}$$

$$J\left(\tfrac{c}{d}, \tfrac{a}{b}\right) = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm i\sqrt{ac}$$

The eigenvalue analysis is inconclusive in this case.

<u>Conserved Quantity:</u>

$$\frac{dy}{dx} = \frac{-cy+dxy}{ax-bxy}$$

$$\Rightarrow \int \frac{a-by}{y}\,dy = \int \frac{-c+dx}{x}\,dx$$

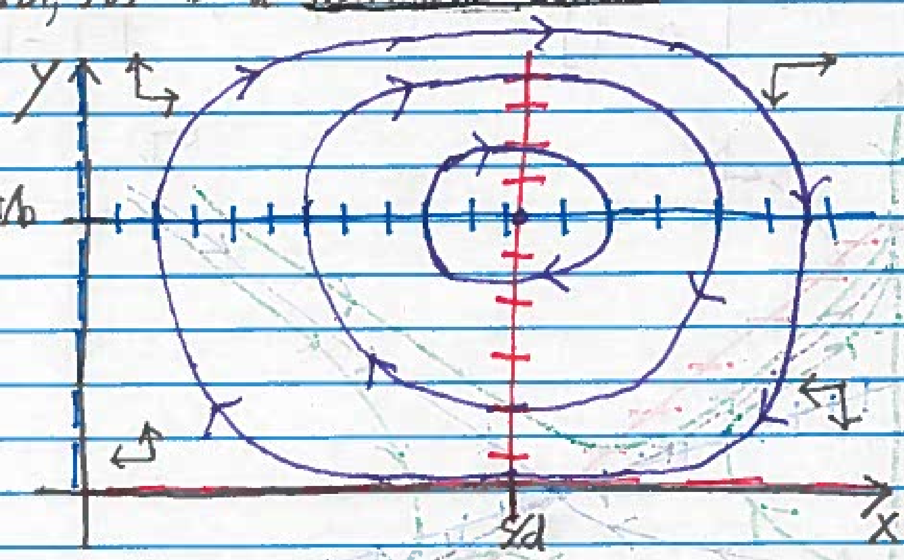
$$\Rightarrow a\ln(|y|) - by = -c\ln(|x|) + dx + C$$

$$\Rightarrow a\ln(|y|) + c\ln(|x|) - by - dx = C$$

The level sets or contours of

$$F(x,y) = a\ln(|y|) + c\ln(|x|) - by - dx$$

correspond to solution curves of the system. This implies $(c/d, a/b)$ is a <u>nonlinear center</u>.



Example:

$\dot{x} = x(3-x) - 2xy$

$\dot{y} = y(2-y) - xy$

Nullclines

N1: $x = 0$ $(\dot{x} = 0)$

N2: $y = -\frac{1}{2}x + \frac{3}{2}$ $(\dot{x} = 0)$

N3: $y = 0$ $(\dot{y} = 0)$

N4: $y = 2 - x$ $(\dot{y} = 0)$

Fixed Points: $(0,0), (3,0), (0,\frac{3}{2}), (1,1)$

$$J = \begin{bmatrix} 3-2x-2y & -2x \\ -y & 2-2y-x \end{bmatrix}$$

$\Rightarrow J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ (unstable node), $J(0,2) = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$ (stable node)

$J(3,0) = \begin{bmatrix} -3 & -4 \\ 0 & -1 \end{bmatrix}$ (stable node), $J(1,1) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$ (saddle)