Lecture 11: Conserved Quantities

Conservative Systems:

Inertial systems of the form:

$$\ddot{x} = F(x).$$

A first integral can be found as follows:

$$\dot{x}\ddot{x} = \dot{x}F(x)$$

$$\frac{1}{2}\frac{d}{dt}(\dot{x}^2) = \frac{dx}{dt}\left(-\frac{dV}{dx}\right)$$

where $V(x) = -\int_{x_0}^{x} F(x)\,dx$. ($x_0$ can be chosen)

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{dV}{dx}\right) = 0$$

For any solution curve there is a constant E such that

$$\boxed{\frac{\dot{x}(t)^2}{2} + V(x(t)) = E}$$

We can also write as a system:

$$\dot{x} = v \qquad \frac{v^2}{2} + V(x) = E$$
$$\dot{v} = F(x)$$

phase portrait $\Longleftrightarrow$ contour plot

Theorem - A conservative system cannot have any attractors or repellors.

~~proof~~:

Suppose there exists (x^*, v^*) that is an attracting fixed point with a basin of attraction A. Then for all $(x_1, y_1) \in A$, $(x_2, y_2) \in A$ it follows that $E(x_1, y_1) = E(x_2, y_2)$ since

$$E(x_1, v_1) = \lim_{t\to\infty} E(x_1(t), v_1(t))$$

$$= E(x^*, v^*)$$

$$= \lim_{t\to\infty} E(x_2(t), v_2(t))$$

$$= E(x_2, v_2)$$

Therefore, E must be constant in entire basin of attraction which we preclude by definition. ■

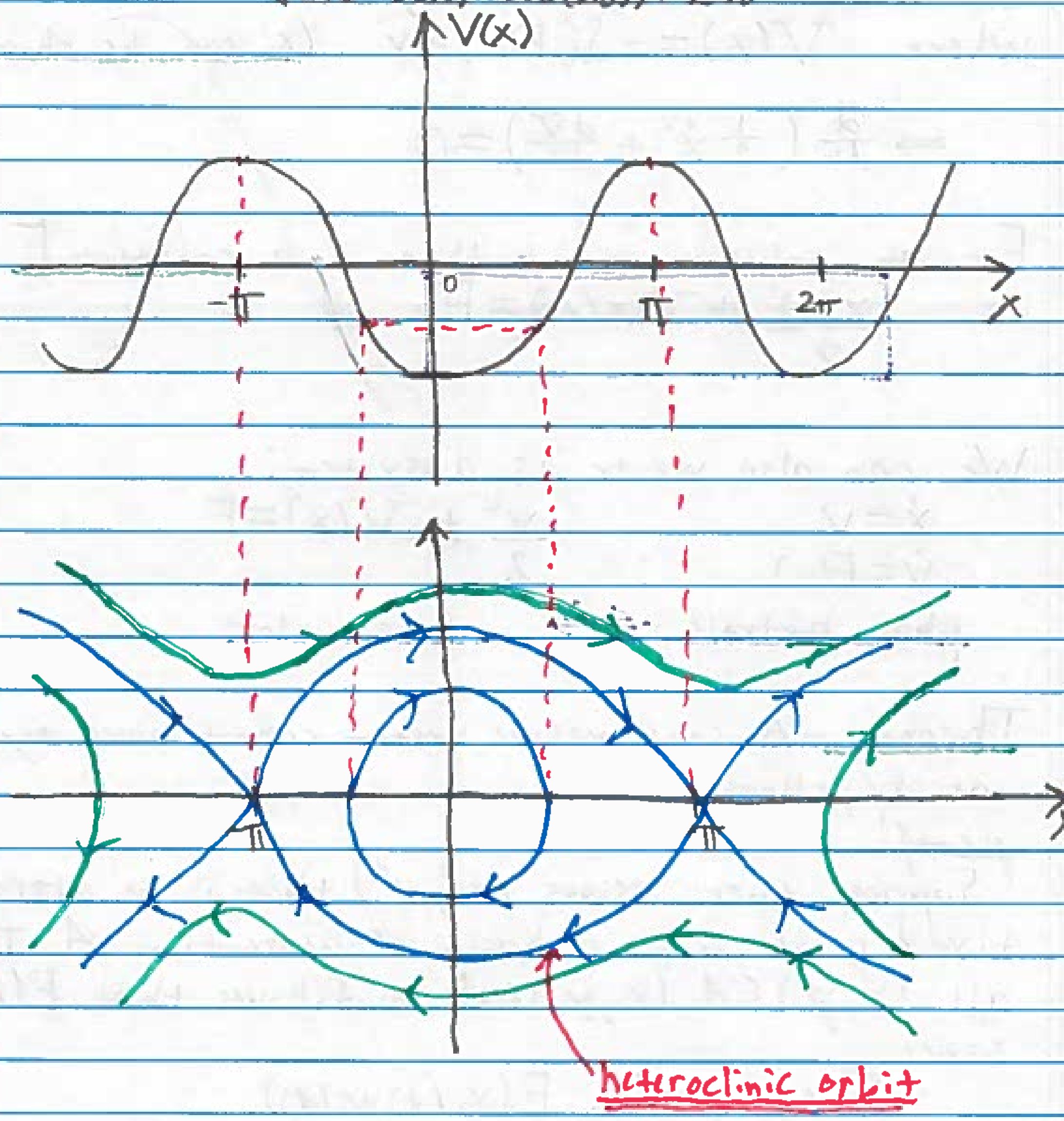
Example:

$$\ddot{x} + \sin(x) = 0$$

$$V(x) = -\cos(x)$$

$$E = \tfrac{1}{2}v^2 - \cos(x)$$

$$\Rightarrow v = \pm\sqrt{2(\cos(x) - \cos(x_0)) + \tfrac{1}{2}v_0^2}$$

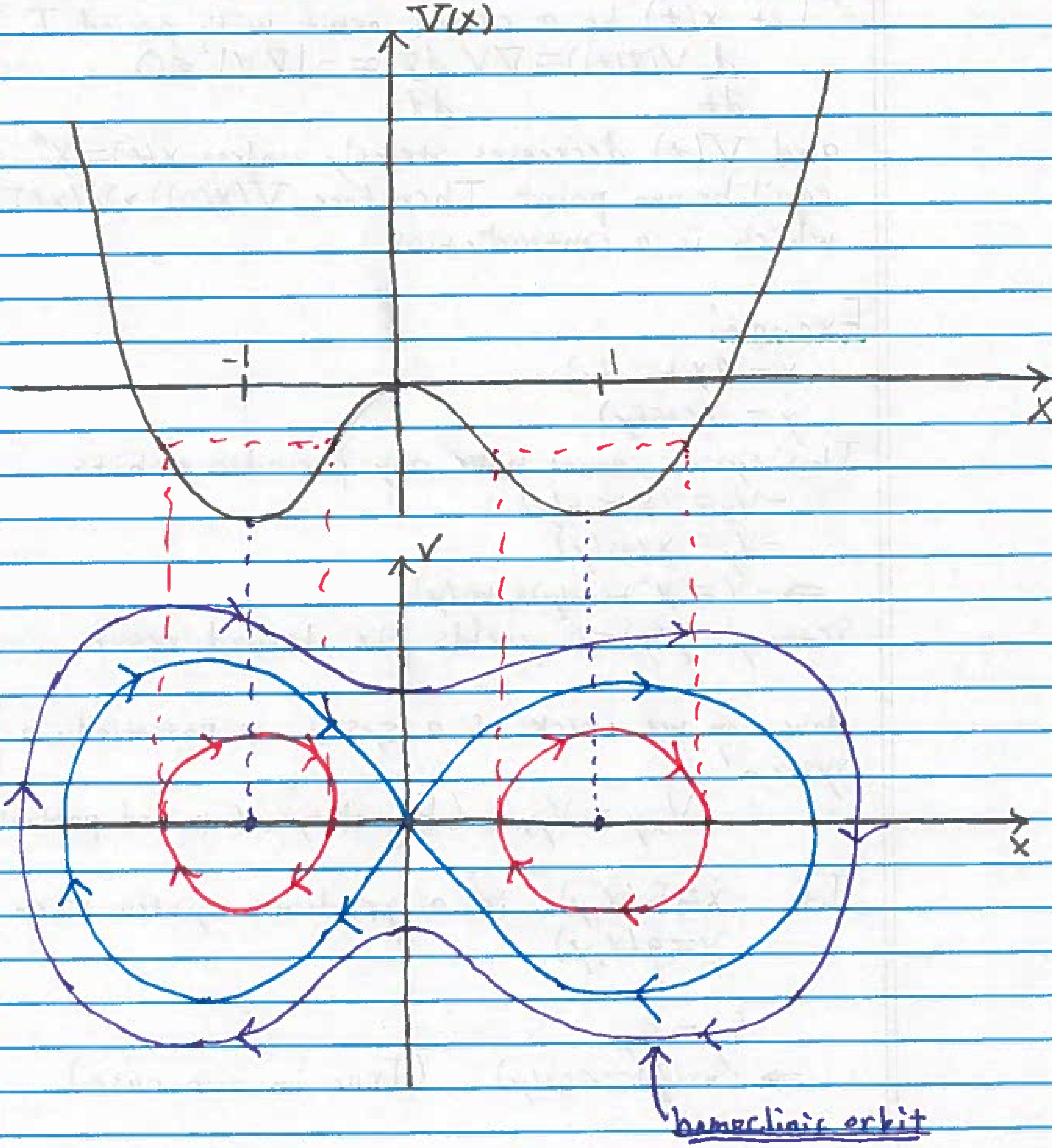


Example:

$\ddot{x} = x - x^3$

$V(x) = \frac{x^4}{4} - \frac{x^2}{2}$

$\Rightarrow E = \frac{1}{2}v^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$



Gradient Systems

$\dot{\vec{x}} = -\nabla V$, where $V: \mathbb{R}^2 \to \mathbb{R}$

Lemma - Gradient systems cannot have closed orbits.

proof:

Let $x(t)$ be a closed orbit with period T. Then,

$$\frac{d}{dt} V(\vec{x}(t)) = \nabla V \cdot \frac{d\vec{x}}{dt} = -|\nabla V|^2 \leq 0$$

and $V(t)$ decreases strictly unless $x(t) = x^*$ is an equilibrium point. Therefore, $V(x(0)) > V(x(T))$ which is a contradiction.

Example:

$\dot{x} = 2x + \sin(y)$

$\dot{y} = x\cos(y)$

This system cannot have any periodic orbits.

$-V_x = 2x + \sin(y)$

$-V_y = x\cos(y)$

$\Rightarrow -V = x^2 + \sin(y)x + g(y)$

Setting $g(y) = 0$ yields the desired result.

How can we check if a system is potentially a gradient system?

$V_{xy} = V_{yx}$ (Equality of mixed partials)

If $\begin{matrix} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{matrix}$ is a gradient system then

$f_y = g_x$

$\Rightarrow \cos(y) = \cos(y)$ (True in our case).

Limit Cycle Example:

$\dot{x} = y$

$\dot{y} = -x + (1-x^2-y^2)y$

Null-clines:

N1: $\dot{x} = 0,\ y = 0$

N2: $\frac{y-y^3}{1+y} = x + x^2$ (Not very informative)

Idea:

$$\ddot{x} + x = (1-x^2-y^2)y$$

$$\Rightarrow \dot{x}\ddot{x} + \dot{x}x = (1-x^2-y^2)\dot{x}^2$$

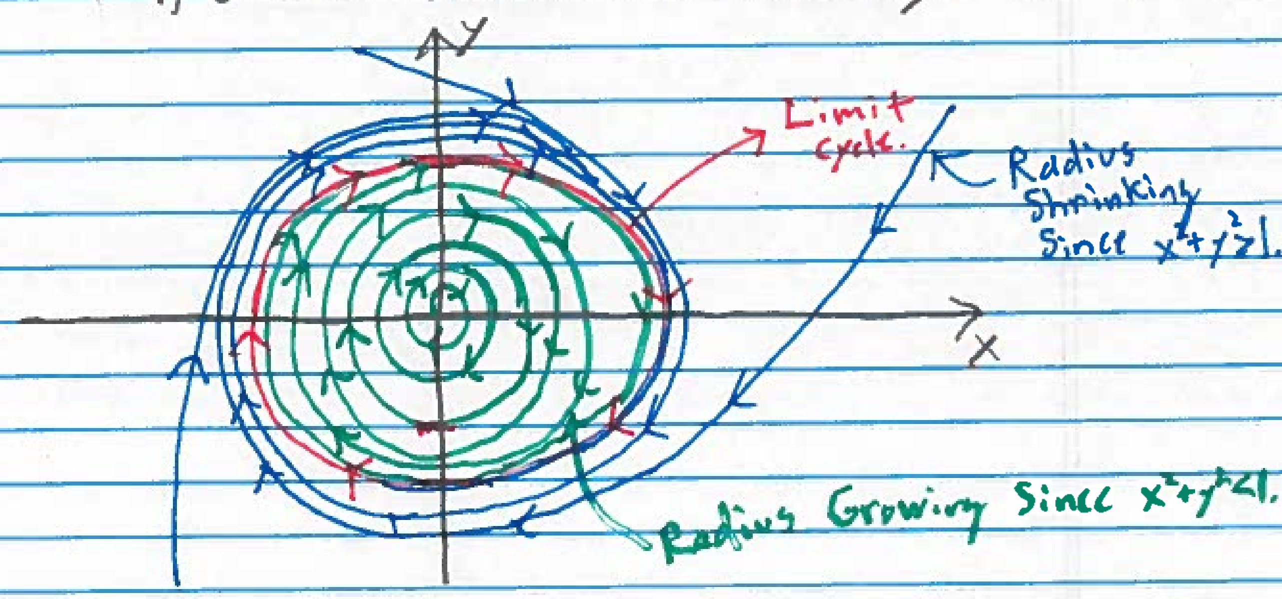
$$\Rightarrow \frac{1}{2}\frac{d}{dt}(\dot{x}^2+x^2) = (1-x^2-y^2)\dot{x}^2$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}(\underbrace{x^2+y^2}_{\text{radius}}) = (1-\underbrace{x^2-y^2}_{\text{radius}})y^2$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}(r^2) = (1-r^2)y^2$$

Cases:

$r<1$, (radial coordinate is growing)
$r=1$, (radial coordinate is constant)
$r>1$, (radial coordinate is decreasing)



Convert to Polar Coordinates:

$$r^2 = x^2 + y^2$$

$$\Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y}$$

$$\Rightarrow \dot{r} = (x\dot{x} + y\dot{y})/r$$

$$= (xy - xy + (1-x^2-y^2)y^2)/r$$

$$= (1-r^2)r\sin^2(\theta)$$

$$\theta = \tan^{-1}(y/x)$$

$$\Rightarrow \dot{\theta} = \frac{1}{1+y^2/x^2} \cdot \frac{x\cdot\dot{y} - y\cdot\dot{x}}{x^2}$$

$$= \frac{x\dot{y} - y\dot{x}}{x^2+y^2}$$

$$= \frac{-x^2 + (1-x^2-y^2)xy - y^2}{x^2+y^2}$$

$$= -1 + (1-r^2)\frac{\sin(2\theta)}{2}$$

$$\Rightarrow \dot{r} = (1-r^2)r\sin^2(2\theta)$$

$$\dot{\theta} = \frac{(1-r^2)\sin(2\theta) - 2}{2}$$

If $r^2 \leq 3$ the motion is clockwise.

Lyapanov Functions

$$\dot{\vec{x}} = F(\vec{x})$$

A continuously differentiable function $L: \mathbb{R}^2 \mapsto \mathbb{R}$ is called a Lyapunov function if $L(x(t))$ strictly decreases along each solution of $\dot{\vec{x}} = F(\vec{x})$ that is not an equilibrium.

Lemma - If $\dot{\vec{x}} = F(\vec{x})$ admits a Lyapanov function, then it cannot have any periodic orbits.

Example:

$$\ddot{x} + \alpha \dot{x} = g(x), \quad \alpha > 0$$

Let $V(x) = -\int_{x_0}^{x} g(x)\,dx$. Then,

$$\frac{1}{2}\frac{d}{dt}(\dot{x})^2 + \alpha \dot{x}^2 = -\frac{d}{dt}(V(x(t)))$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + V(x(t))\right) = -\alpha \dot{x}^2 < 0$$

The function $L(v, x) = \frac{1}{2}v^2 + V(x)$ is a Lyapunov function.

Summary:

1. $\ddot{x} = -\frac{dV}{dx} \longrightarrow$ Conservative, $E(x, \dot{x})$ is conserved } Many periodic orbits
2. $\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} = -\nabla V \rightarrow$ Gradient system, V decreases along solutions
3. $\ddot{x} + \alpha\dot{x} = -\frac{dV}{dx} \longrightarrow E(x, \dot{x})$ decreases.

} No periodic solutions.