Lecture 12: Limit Cycles

Definition - A closed trajectory is a limit cycle if it is seperated from all other closed trajectories.

a.) A limit cycle is stable if there is a tubular neighborhood such that trajectories that enter the neighborhood approach the limit cycle as $t \to \infty$.

b.) A limit cycle is unstable if it is not stable.

Poincare-Bendixon Theorem - Consider $\dot{\vec{x}} = F(\vec{x})$, with F continuously differentiable. Assume $R \subset \mathbb{R}$ is closed and bounded.

(i) R does not contain any fixed points.

(ii) There exists $\vec{x}(0) \in R$ so that $\vec{x} \in R$ for all $t \geq 0$.

Then R contains a limit cycle.

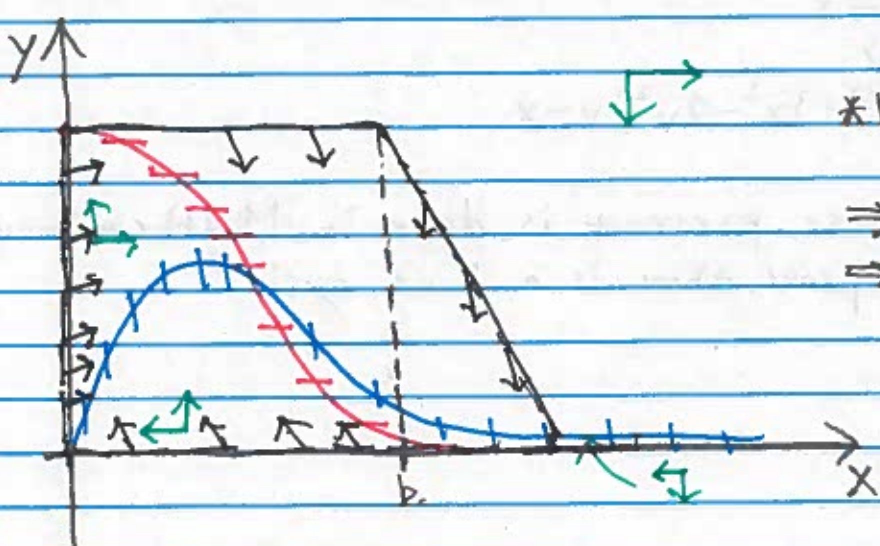
Example:

$$\dot{x} = -x + ay + x^2y, \quad a, b > 0.$$
$$\dot{y} = b - ay - x^2y$$

Null-clines

$$y = \frac{x}{a+x^2}, \quad \dot{x} = 0$$

$$y = \frac{b}{a+x^2}, \quad \dot{y} = 0$$



*When is $\frac{dy}{dx} < -1$?

$\Rightarrow b - ay - x^2y < +x - ay - x^2y$

$\Rightarrow x > b$

We now check the stability of the fixed point

$$x^* = b, \quad y^* = \frac{b}{a+b^2}$$

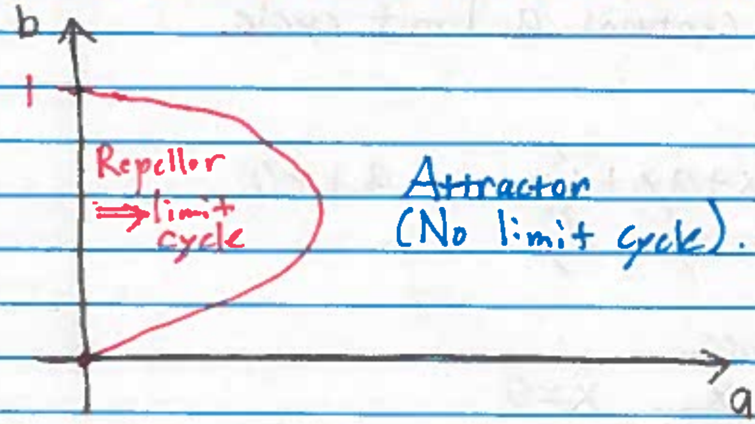
$$\Rightarrow J = \begin{bmatrix} -1+2xy & a+x^2 \\ -2xy & -a-x^2 \end{bmatrix}$$

$$J(x^*, y^*) = \begin{bmatrix} -1+\frac{2b}{a+b^2} & a+b^2 \\ \frac{-2b}{a+b^2} & -a-b^2 \end{bmatrix}$$

$$\det(A) = a+b^2 > 0$$

$$Tr(A) = -1-a-b^2+\frac{2b^2}{a+b^2}$$

$$Tr(A) = 0 \Leftrightarrow a^2 + a(2b^2+1) + b^2(b^2-1) = 0$$



Example:

$$\ddot{x} = (1-3x^2-2\dot{x}^2)\dot{x} - x$$

Let $v = \dot{x}$

$$\dot{x} = v$$

$$\dot{v} = (1-3x^2-2v^2)v - x$$

The phase portrait is done in Mathematica. How can we prove there is a limit cycle?

Convert to polar coordinates:

$$\dot{r} = \cos\theta\,\dot{x} + \sin\theta\,\dot{v}$$

$$\Rightarrow \dot{r} = \cos\theta\, v + \sin\theta[(1-3x^2-2v^2)v - x]$$

$$\Rightarrow \dot{r} = r\cos\theta\sin\theta + \sin\theta[(1-3r^2\cos^2\theta-2r^2\sin^2\theta)\cdot r\sin\theta - r\cos\theta]$$

$$\Rightarrow \dot{r} = r\sin^2\theta[1-2r^2-r^2\cos^2\theta]$$

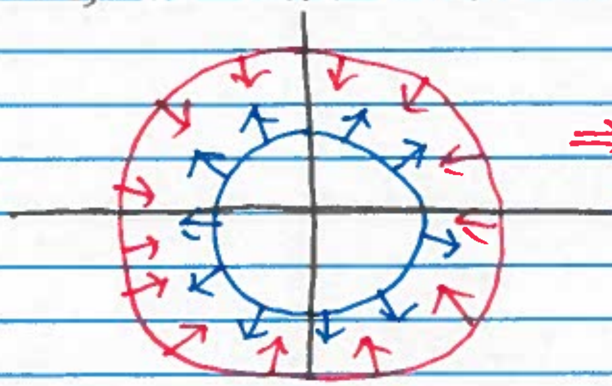
<u>Let $r = 1/2$</u>

Then $\dot{r} = \frac{1}{2}\sin^2\theta\,[1-\frac{1}{2}-\frac{1}{2}\cos^2\theta] \geq 0$

<u>Let $r = 1/\sqrt{2}$</u>

Then $\dot{r} = [-\frac{1}{2}\cos^2\theta] \leq 0$

Therefore, we have constructed a trapping region



$\Rightarrow$ A limit cycle exists!!