Lecture 13: Index Theory

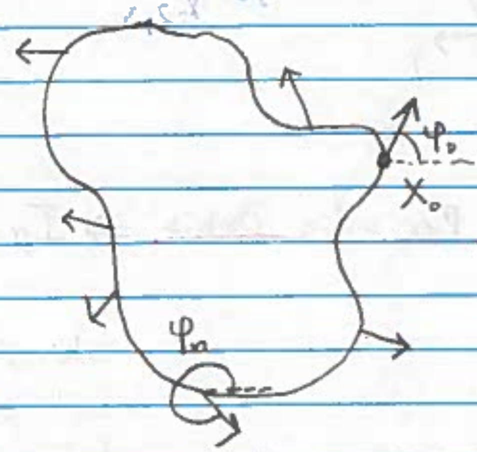
How can we be sure no periodic orbit exists?

Consider,

$$\dot{\vec{x}} = F(\vec{x})$$

with $F: \mathbb{R}^2 \to \mathbb{R}^2$ continuously differentiable.

Take a closed curve Γ with no self intersections, that does not pass through a fixed point.



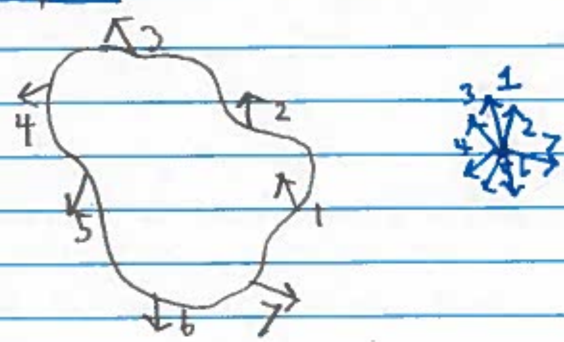
1. Start at x_0, traverse Γ counterclockwise and take angle φ of $F(\vec{x})$. This angle changes continuously as Γ is traversed.
2. After one pass we again end up at x_0 with an angle $\varphi_f = \varphi_0 + 2\pi n$, $n \in \mathbb{Z}$

Define - $I_\Gamma = \frac{1}{2\pi}(\varphi_f - \varphi_0)$

Index of a curve.

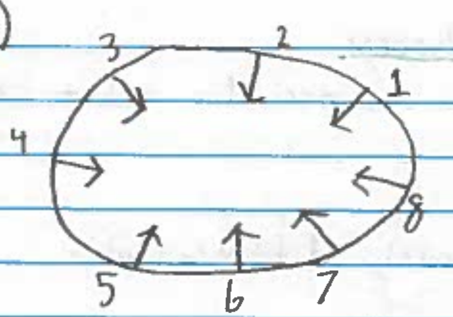
Examples:

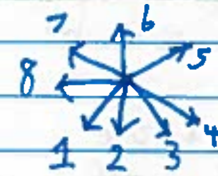
a.)



$I_\Gamma = 1$.

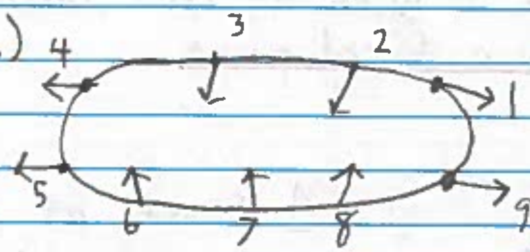
b.)

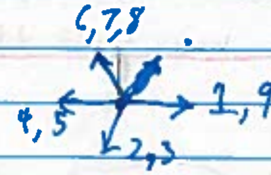




$I_p = 1.$

c.)



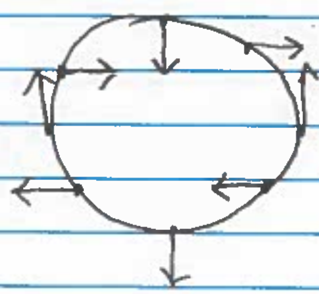


$I_p = -1.$

d.)

Periodic Orbit $\Rightarrow I_p = 1.$

e.) $\begin{cases} \dot{x} = x^2 y \\ \dot{y} = x^2 - y^2 \end{cases}$, Γ = unit circle



$I_p = 0$

Properties of the Index

1. If Γ can be deformed continuously into $\tilde{\Gamma}$ without passing through any equilibrium points then
$$I_\Gamma = I_{\tilde{\Gamma}}$$
proof:
I_Γ varies continuously as Γ is deformed, but I_Γ is integer valued.

2. If Γ does not contain any fixed points then $I_\Gamma = 0$.
proof:
Property 1 implies we can shrink Γ to a point with changing the index.

3. If we replace $F(\vec{x})$ by $F(-\vec{x})$ the index is not changed.
proof:
Each angle is replaced by $\varphi + \pi$, hence $\varphi_f - \varphi_0$ is the same.

4. The index of a periodic orbit is one.

5. If Γ is continuously deformed with creating any fixed points on Γ, I_Γ stays the same.

Theorem - Assume F is continuously differentiable. Inside each periodic orbit, there is at least one equilibrium.
proof:
Follows from items 2 and 4.

Index of isolated fixed point: Let x^* be an isolated fixed point of $\dot{\vec{x}} = F(\vec{x})$. Define
$I(\vec{x}^*)$ = index of simple closed curve that encloses x^* and no other fixed points.

$I(\vec{x}^*)$ is well defined by property 1.

Consequences

1. If $\vec{x}^*$ is an attractor or repellor then $I(\vec{x}^*)=1$
2. If $\vec{x}^*$ is a saddle then $I(x^*)=-1$.

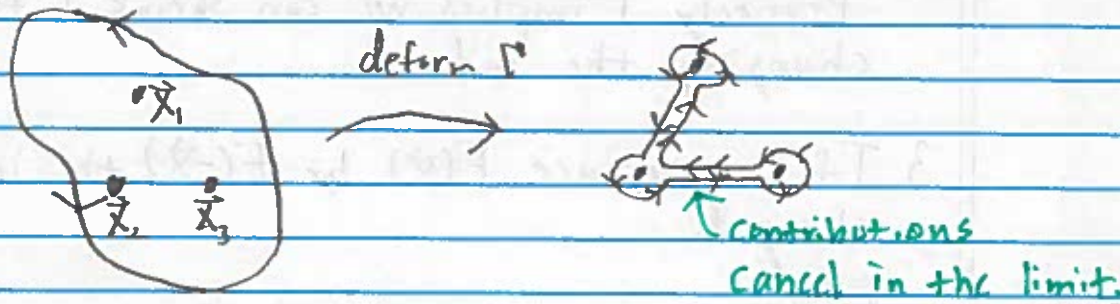
proof:

Follows from examples b and c and properties 1, 3, 5.

Theorem - If Γ is a closed simple curve that contains n isolated fixed $\vec{x}_1, \ldots, \vec{x}_n$ then

$$I_\Gamma = I(\vec{x}_1) + \ldots + I(\vec{x}_n)$$

proof:



Corollary: A periodic orbit must enclose whose indices sum to $+1$!