Lecture 13: Index Theory

How can we be sure no periodic orbit exists?
Consider,
\[ \dot{x} = F(x) \]
with \( F: \mathbb{R}^2 \to \mathbb{R}^2 \) continuously differentiable.

Take a closed curve \( \Gamma \) with no self-intersections, that does not pass through a fixed point.

1. Start at \( x_0 \), traverse \( \Gamma \) counterclockwise and take angle \( \psi \) of \( F(x) \). This angle changes continuously as \( \Gamma \) is traversed.
2. After one pass we again end up at \( x_0 \) with an angle \( \psi_f = \psi_0 + 2\pi n \), \( n \in \mathbb{Z} \).

Define \( I_\Gamma = \frac{1}{2\pi} (\psi_f - \psi_0) \)

Index of a curve.

Examples:

a. \[ I_\Gamma = 1. \]
b.)

\[ \Gamma = \begin{cases} \Gamma_1 = 1 \end{cases} \]

\[ \Gamma \in \text{unit circle} \]

\[ \Gamma_\rho = 0 \]

d.)

\[ \text{Periodic Orbit} \Rightarrow \Gamma_\rho = 1 \]

e.)

\[ \begin{cases} \dot{x} = x^2 \\ \dot{y} = x^2 - y^2 \end{cases} \]

\[ \Gamma = \text{unit circle} \]

\[ \Gamma_\rho = 0 \]
Properties of the Index

1. If $\gamma$ can be deformed continuously into $\tilde{\gamma}$ without passing through any equilibrium points then $I_{\gamma} = I_{\tilde{\gamma}}$

   \textit{proof:}
   
   $I_{\gamma}$ varies continuously as $\gamma$ is deformed, but $I_{\gamma}$ is integer valued.

2. If $\gamma$ does not contain any fixed points then $I_{\gamma} = 0$.

   \textit{proof:}
   
   Property 1 implies we can shrink $\gamma$ to a point with changing the index.

3. If we replace $F(x)$ by $F(-x)$ the index is not changed.

   \textit{proof:}
   
   Each angle is replaced by $\theta + \pi$, hence $\Psi - \Psi_0$ is the same.

4. The index of a periodic orbit is one.

5. If $\gamma$ is continuously deformed with creating any fixed points on $\gamma$, $I_{\gamma}$ stays the same.

\textbf{Theorem} - Assume $F$ is continuously differentiable. Inside each periodic orbit, there is at least one equilibrium.

\textit{proof:}

Follows from items 2 and 4.

\textbf{Index of isolated fixed point:} Let $x^*$ be an isolated fixed point at $\dot{x} = F(x)$. Define

$I(x^*) =$ index of simple closed curve that encloses $x^*$ and no other fixed points.

$I(x^*)$ is well defined by property 1.
Consequences

1. If $x^*$ is an attractor or repellor then $I(x^*) = 1$.
2. If $x^*$ is a saddle then $I(x^*) = -1$.

Proof:
Follows from examples b and c and properties 1, 3, 5.

Theorem: If $\Gamma$ is a closed simple curve that contains $n$ isolated fixed $x_1, \ldots, x_n$ then

$$I_\Gamma = I(x_1) + \ldots + I(x_n)$$

Proof:
Deform $\Gamma$

Contributions cancel in the limit.

Corollary: A periodic orbit must enclose whose indices sum to $+1$. 