Lecture 14: Iterated Maps

$f: \mathbb{R} \to \mathbb{R}$, $f: [0,1] \to [0,1]$

Discrete Dynamical System

$$x_{n+1} = f(x_n)$$

Newton's Method

Find root x^* of $g(x)$:

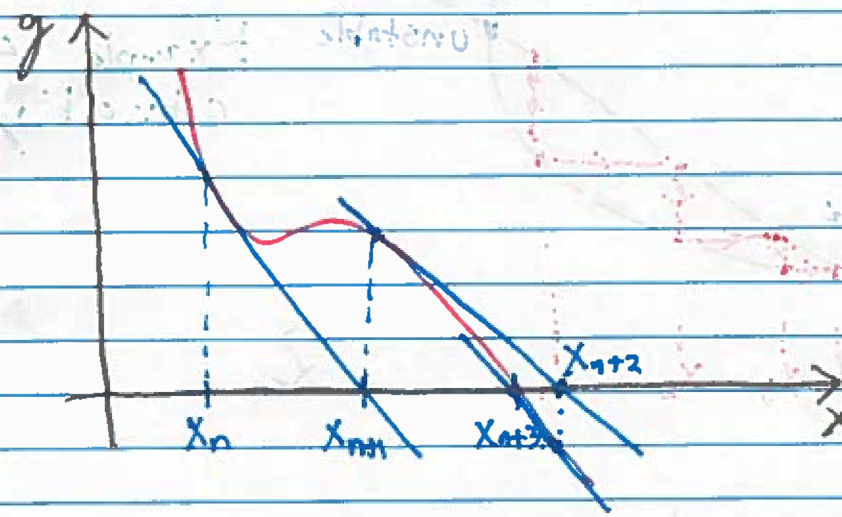
Given x_n find a better approximation of x^*.

Taylor expand:

$$0 = g(x) = g(x_n) + g'(x_n)(x - x_n)$$

Solve for x:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$



Definitions-

1. Orbit of x_0: $\gamma(x_0) = \{x_n : x_n = f(x_{n-1})\}$.
2. Fixed points: x^* is a fixed point if $x^* = f(x^*) \Rightarrow \gamma(x^*) = \{x^*\}$
3. Period k orbits: $\gamma(x_0) = \{x_0, x_1, \ldots, x_k\}$.

Stability:

Let x^* be a fixed point of f and assume that f is differentiable near x^*. Then,

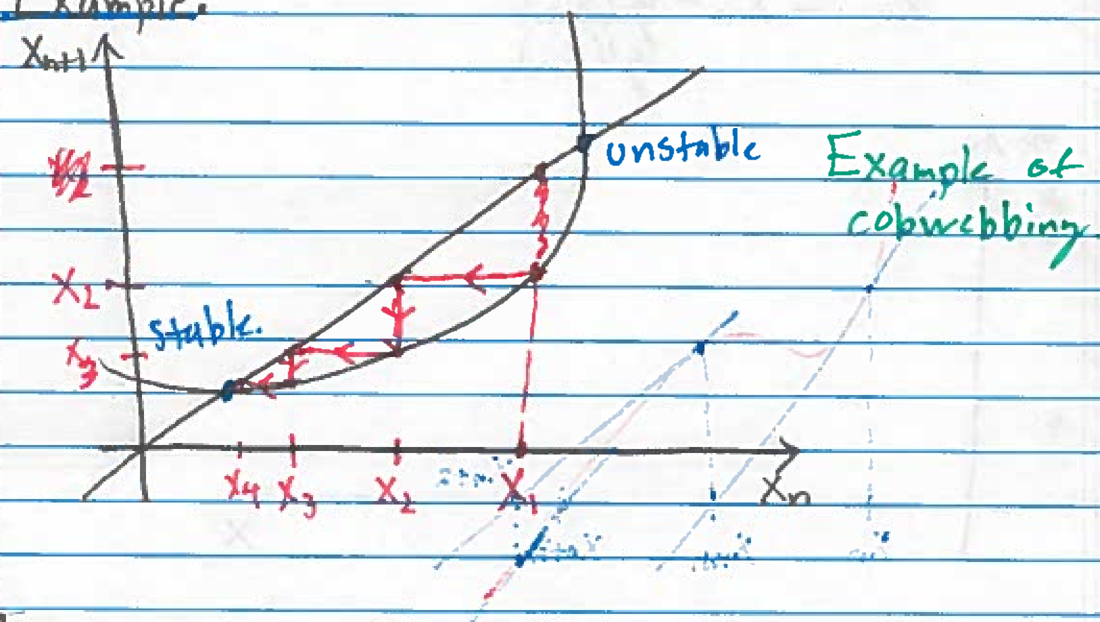
$$x^* \text{ is } \begin{cases} \text{stable if } |f'(x^*)| < 1 \\ \text{unstable if } |f'(x^*)| > 1. \end{cases}$$

~~proof~~:

Let δ_n be the sequence satisfying $\delta_n = x_n - x^*$ for $x_0 = x^* + \delta_0$. Then,

$$x_1 = f(x^* + \delta_0) = f(x^*) + f'(x^*)\delta_0 + O(\delta_0^2)$$
$$\Rightarrow \delta_1 = f'(x^*)\delta_0 + O(\delta_0^2)$$
$$\Rightarrow \delta_2 \approx f'(x^*)\delta_1 \approx f'(x^*)^2 \delta_0$$
$$\vdots$$
$$\Rightarrow \delta_n \approx f'(x^*)^n \delta_0$$
$$\Rightarrow \lim_{n\to\infty} |\delta_n| = \begin{cases} 0 & \text{if } |f'(x^*)|<1 \\ \infty & \text{if } |f'(x^*)|>1 \end{cases}$$

Example:



Logistic Equation:

$$x_{n+1} = r x_n (1-x_n), \quad \text{assume } 0<r<4$$

Discrete model of population growth.

Fixed points satisfy:

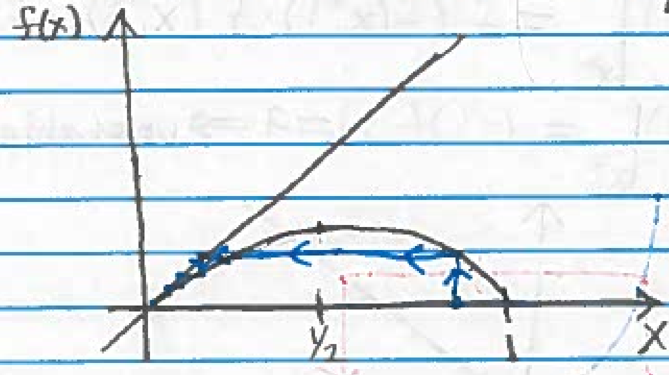
$$x^* = r x^* (1-x^*) = f(x)$$
$$\Rightarrow x^*(1 - r(1-x^*)) = 0$$
$$\Rightarrow x^* = 0, \quad x^* = 1 - 1/r.$$

Extinction — Robust survival, exists if $r>1$.

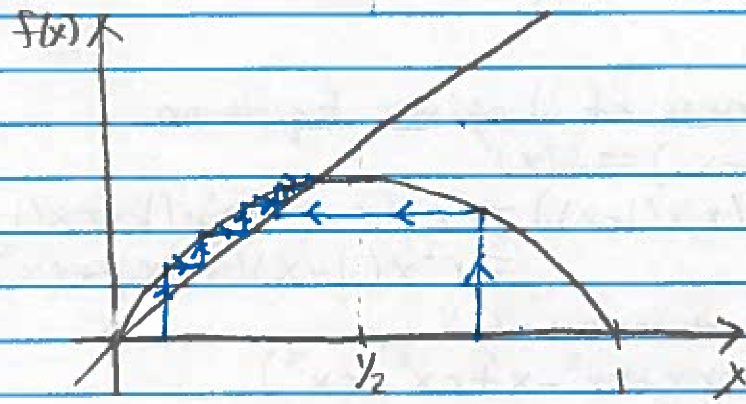
$$f'(x) = r(1-2x)$$
$$\Rightarrow f'(0) = r$$
$$f'(1-1/r) = r\left(\tfrac{2}{r} - 1\right) = 2 - r.$$

Therefore,

1. If $0<r<1$ then
   0 is stable (the species goes extinct)



2. If $1<r<3$ then
   0 is unstable
   $1-1/r$ is stable



What happens if $3<r<4$?

<u>Period Two Orbits:</u>

$f(x) = -x^3$

Fixed Points:

$x^* + x^{*3} = 0$

$\Rightarrow x^* = 0$

$f'(0) = 0 \Rightarrow$ stable.

Period 2 orbits satisfy

$f^2(x) = (f \circ f)(x) = x^{*9} = x^*$

$\Rightarrow x^* = 0, \; x^*_{1,2} = \pm 1$

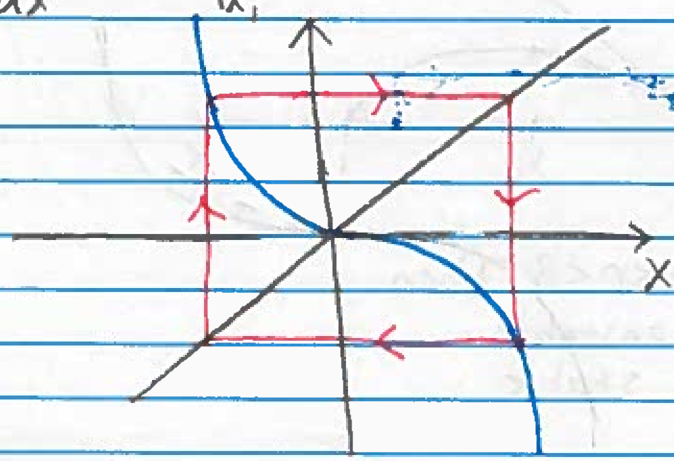
Not really 2-orbit   This is the real 2-orbit

Stability of 2-orbit

$$\frac{d}{dx} f(f(x)) = f'(f(x))\cdot f'(x)$$

$$\Rightarrow \frac{d}{dx} f(f(x))\Big|_{x_1^*} = f'(f(x_1^*))\cdot f'(x_1^*) = \overbrace{f'(x_2^*)f'(x_1^*)}^{\text{Nice trick!}}$$

$$\Rightarrow \frac{d}{dx} f(f(x))\Big|_{x_1^*} = (-3)(-3) = 9 \Rightarrow \text{unstable}$$



Period Two Orbits of Logistic Equation

$$x_{n+1} = r x_n(1-x_n) = f(x)$$

$$f(f(x)) = f(rx(1-x)) = r\cdot(rx(1-x))\cdot(1-rx(1-x))$$

$$= r^2 x(1-x)(1-rx+rx^2)$$

Period two orbits satisfy:

$$1 = r^2(1-rx+rx^2-x+rx^2-rx^3)$$

$$\Rightarrow r^3x^3 - 2r^3x^2 + (1+r)r^2x - r^2 + 1 = 0$$

To find roots use long division. (See Mathematica).