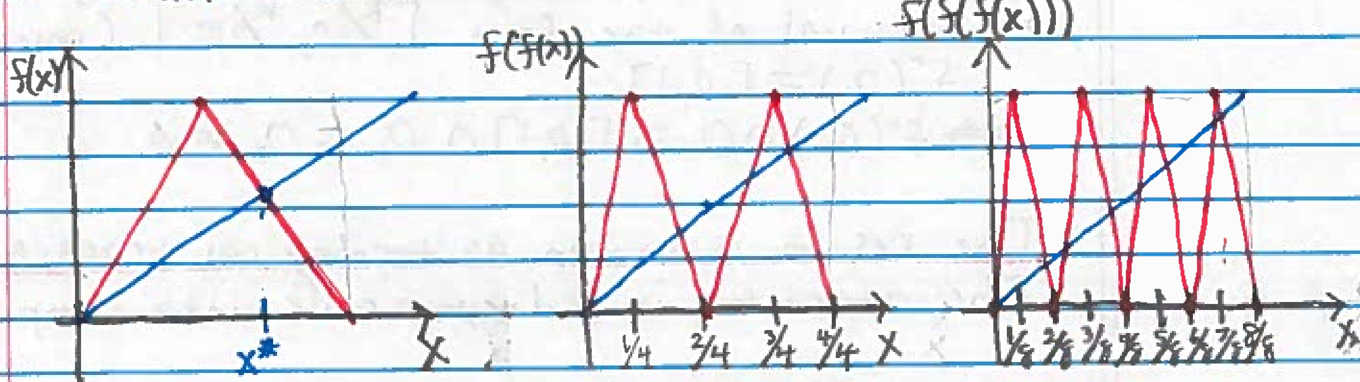
Lecture 15: Chaotic Dynamics

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2-2x & 1/2 < x \le 1 \end{cases}$$

$$x_{n+1} = f(x_n)$$



Fixed Points: $x^* = 0$, $x^* = 2/3$

Period Two Orbits: $\gamma = \{2/5, 4/5\}$.

Period Three Orbits: $\gamma = \{2/9, 4/9, 8/9\}$, $\gamma = \{2/7, 4/7, 6/7\}$.

⋮

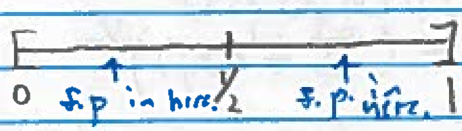
Theorem - Periodic Orbits are dense in $[0,1]$

proof:

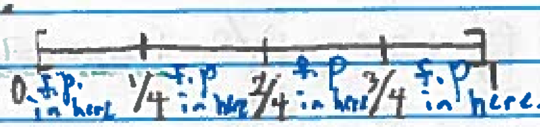
Let $x \in [0,1]$. We want to show for all $\varepsilon > 0$ there exists $x^* \in [0,1]$ and $n \in \mathbb{N}$ such that $f^n(x^*) = x^*$ and $|x^* - x| < \varepsilon$. We know that f^n maps each interval $[k/2^n, k+1/2^n]$ into for $k \in \mathbb{N}$. As a result each interval $[k/2^n, k+1/2^n]$ contains a fixed point of f^n.

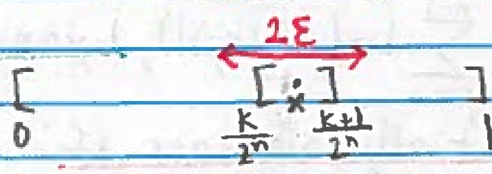
Graphical Interpretation:

n=1:



n=2:





$\Rightarrow$ Pick $1/2^n < 2\varepsilon$ and k so that $k/2^n < x < k+1/2^n$.

Theorem - For all open intervals $O_1, O_2 \subset [0,1]$ there exists $n \in \mathbb{N}$ such that $f^n(O_1) \cap O_2 \neq \emptyset$.

proof:

For n sufficiently large and for some K, O_1 contains an interval of the form $[K/2^n, K/2^{n+1}]$. Consequently,

$f^n(O_1) = [0,1]$

$\Rightarrow f^n(O_1) \cap O_2 = [0,1] \cap O_2 = O_2 \neq \emptyset$.

* The result is known as topological transitivity - The attractor is indecomposable into components.

Lyapunov Exponents:

How to measure sensitivity to initial data?

1. Pick x_0 and compute $\gamma(x_0) = \{x_0, x_1, x_2, \ldots\}$
2. Pick y_0 close to x_0 and compute $\gamma(y_0) = \{y_0, y_1, y_2, \ldots\}$.
3. Let $\delta_n = y_n - x_n$

$$\Rightarrow \delta_n = f(y_{n-1}) - f(x_{n-1})$$
$$= f(x_{n-1} + \delta_{n-1}) - f(x_{n-1})$$
$$= \frac{f(x_{n-1} + \delta_{n-1}) - f(x_{n-1})}{\delta_{n-1}} \cdot \delta_{n-1}$$
$$\approx f'(x_{n-1})\,\delta_{n-1}$$
$$\Rightarrow \delta_n \approx f'(x_{n-1}) \cdots f'(x_0)\,\delta_0$$
$$= \left(\prod_{j=0}^{n-1} f'(x_j)\right)\delta_0$$

We expect

$$|\delta_n| = |\delta_0| \exp(n\lambda(x_0))$$
$$\Rightarrow \exp(\lambda(x_0)) = \left(\prod_{j=0}^{n-1} f'(x_j)\right)^{1/n}.$$

We define:

$$L(x_0) = \lim_{n\to\infty}\left(\prod_{j=0}^{n-1} |f'(x_j)|\right)^{1/n}, \text{ Lyapunov multiplier}$$

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \ln(|f'(x_j)|), \text{ Lyapunov exponent.}$$

* Orbits exponentially seperate if:

$L(x_0) > 0$ or $\lambda(x_0) > 0$.

Definition - We say $x_{n+1}=f(x_n)$ is chaotic if

1. It has sensitive dependence on initial conditions
2. The set of periodic orbits is dense.
3. It is topologically transitive

Example:

$$x_{n+1} = \begin{cases} 2x_n & 0 \le x \le \frac{1}{2} \\ 2-2x_n & \frac{1}{2} < x \le 0 \end{cases}$$

This system is chaotic.

Example:

$x_{n+1} = 4x_n(1-x_n)$

This system is chaotic.

proof:

Let $f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2-2x, & \frac{1}{2} < x \le 1 \end{cases}$ and $g(x) = 4x(1-x)$.

Define,

$$h(x) = \frac{1}{2}(1-\cos(2\pi x)). \quad \text{(Onto map).}$$

$$\begin{aligned}
\Rightarrow h(f(x)) &= \tfrac{1}{2}(1-\cos(4\pi x)) \\
&= \tfrac{1}{2} - \tfrac{1}{2}(2\cos^2(2\pi x)-1) \\
&= 1 - \cos^2(2\pi x) \\
&= (1-\cos(2\pi x))(1+\cos(2\pi x)) \\
&= 4 \cdot \tfrac{1}{2}(1-\cos(2\pi x))\tfrac{1}{2}(1+\cos(2\pi x)) \\
&= 4 \cdot h(x)(1-h(x)) \\
&= g(h(x))
\end{aligned}$$

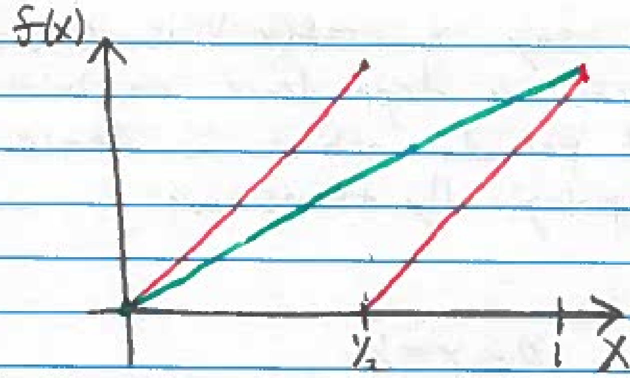
$\Rightarrow f = h^{-1} \circ g \circ h$

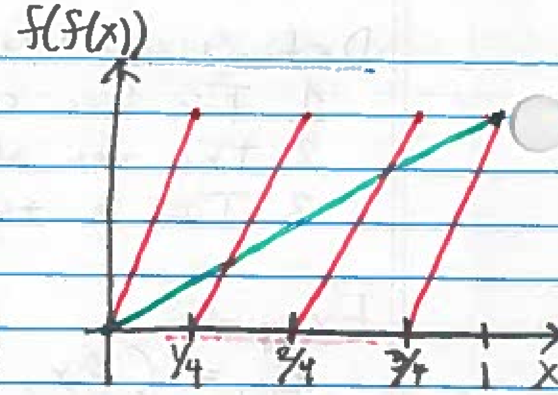
$\Rightarrow f$ and g have the same properties as functions. ■

Example:

$$x_{n+1} = \begin{cases} 2x_n & 0 \le x \le \frac{1}{2} \\ 2x_n - 1 & \frac{1}{2} < x \le 1 \end{cases}$$

$= 2x \bmod (1).$





Fixed Points: 0, 1

Period Two Orbits: $\{1/3, 2/3\}$

Period Three Orbits: $\{1/7, 2/7, 4/7\}$, $\{3/7, 6/7, 5/7\}$.

This map is chaotic. An easy way to think about this problem is to convert to binary:

$1 = .11111\ldots$
$0 = .00000\ldots$ } fixed points

$1/3 = .010101\ldots$ period two orbit.

$1/7 = .001001001\ldots$
$3/7 = .011011011\ldots$ } period three orbit.

Graphical interpretation of $f(x) = 2x \bmod (1)$.