Lecture 16: Fractals

Dimension - How do we measure the dimension of a set? One idea is to count the number of coordinates needed to describe a set.

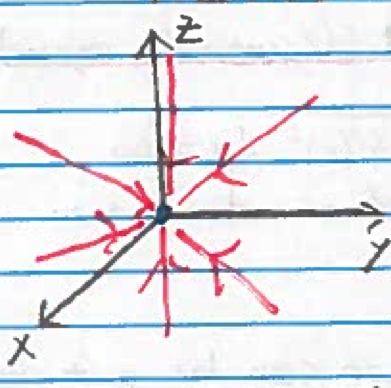
Example:

- Interior of sphere is three dimensional. Coordinates are radial distance, latitude, and longitude.
- Boundary of sphere is two dimensional. Coordinates are latitude and longitude.
- The equator is one dimensional. Longitude is a a coordinate.

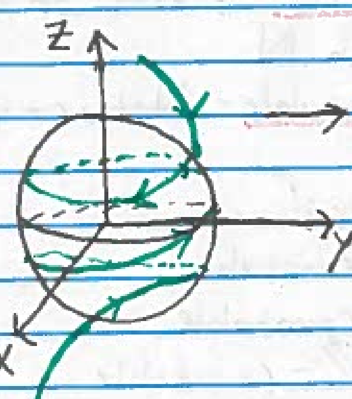
Example:

Dynamics in spherical coordinates

1. $\dot{\rho} = -\rho$, $\dot{\theta} = 1$, $\dot{\phi} = 1$ $\longrightarrow$ Attracting set is $(0,0,0)$, 0 dimensional manifold.



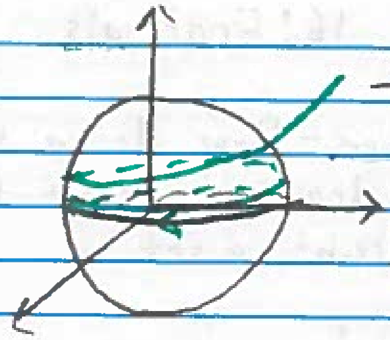
2. $\dot{\rho} = \rho(1-\rho)$, $\dot{\theta} = \sqrt{2}$, $\dot{\phi} = 1$



$\longrightarrow$ Attracting set is the sphere $\rho = 1$, 2 dimensional manifold.

*The dynamics on the sphere in this case is known as quasiperiodicity.

3. $\dot{\rho} = \rho(1-\rho)$
$\dot{\theta} = 1$
$\dot{\phi} = \pi/2 - \phi$



→ Attracting set is the equator $\rho=1, \phi=\pi/2$, a 1 dimensional manifold

Example:

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi k^2 x)}{\pi k^2}$$, on the interval $[0,1]$

$$|f(x)| \leq \sum_{k=1}^{\infty} \frac{1}{\pi k^2} = \frac{1}{6} \Rightarrow f \text{ is continuous}$$

However:

$$f'(x) = \sum_{k=1}^{\infty} \cos(\pi k^2 x)$$

$\Rightarrow$ For large K $\cos(\pi k^2 x) \approx$ i.o.
$\Rightarrow |f'(x)| = \infty$
$\Rightarrow$ f is not differentiable anywhere.

Consequence:

$$L = \int_0^1 \sqrt{1 + f'(x)^2}\, dx = \infty$$

*we cannot define dimension in the classical sense.

Size of Sets:

Countable - Finite or can be put in one-to-one correspondence with $\mathbb{N}$.

Uncountable - Not countable.

Examples:

$\mathbb{Z}$ - countable
$\mathbb{Q}$ - countable
$\mathbb{Z} \times \mathbb{Z}$ - countable
$\{0,1\} \times \{0,1\}$ - countable
$[0,1]$ - uncountable
$\{0,1\} \times \{0,1\} \times \{0,1\} \times \ldots$ - uncountable

Sets of Measure 0:

A set S has measure 0 if $\forall \varepsilon > 0$, S is a subset of a union of open cubes the sum of whose volumes is less than ε.

Example:

$\mathbb{Q}$ is a set of measure 0.

proof:

We can index $\mathbb{Q}$ by points $\{r_1, r_2, \ldots\}$. Let

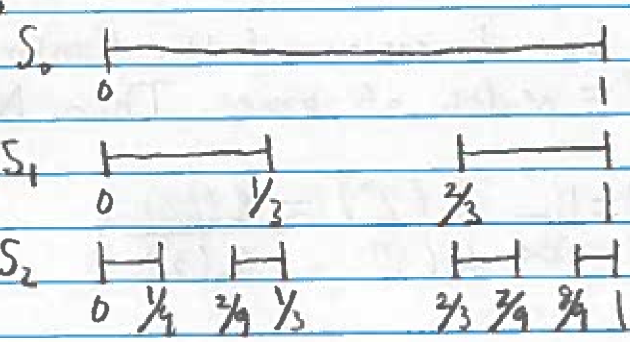
$$b_i = \{r_i - \varepsilon/2\cdot 2^i,\ r_i + \varepsilon/2\cdot 2^i\}.$$

Then,

$$V\left(\bigcup_{i=1}^{\infty} b_i\right) \leq \sum_{i=1}^{\infty} V(b_i) = \sum_{i=1}^{\infty} \varepsilon/2^i \leq \pi/6 \cdot \varepsilon.$$

Example:

The Cantor set is found by removing middle third of sets:

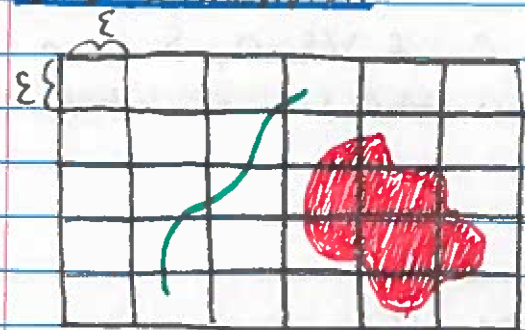


$\vdots$

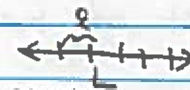
$$S_\infty = \bigcap_{n=1}^{\infty} S_n$$

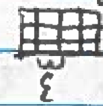
1. S_∞ is uncountable – Can be put into 1-1 correspondence with $[0,1]$ by binary representation.
2. S_∞ has measure 0 – Cover by balls of volume $(1/3)^n \cdot 2^n$ and take limit $n \to \infty$.

Box Dimension



Let $A \subset \mathbb{R}^n$. Take a mesh of boxes of length ε. Let $N(\varepsilon)$ be the number of boxes that intersect with A.

1-dim: $N(\varepsilon) \sim 1/\varepsilon$  $N = L/\varepsilon$

2-dim: $N(\varepsilon) \sim 1/\varepsilon^2$  $N = A/\varepsilon^2$

Assuming some scaling law

$$N(\varepsilon) \sim 1/\varepsilon^d$$

then

$$\boxed{d = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)}, \quad d\text{-box dimension}}$$

Example:

What is the box dimension of the Cantor set?

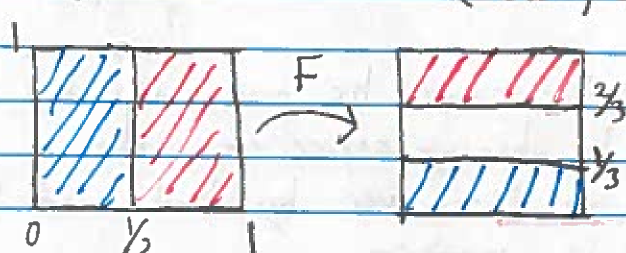
Let $\varepsilon_n = (1/3)^n$ = width of boxes. Then $N(\varepsilon) = 2^n$.

Consequently,

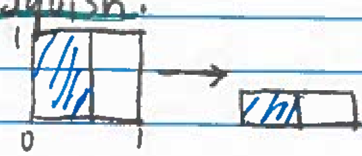
$$d = \lim_{n \to \infty} \frac{\ln(2^n)}{\ln(3^n)} = \frac{\ln(2)}{\ln(3)}.$$

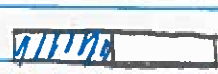
Example:

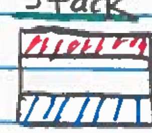
$$(x_{n+1}, y_{n+1}) = F(x_n, y_n) = \begin{cases} (2x_n, y_n/3) & 0 \le x \le 1/2 \\ (2x_n - 1, y_n/3 + 2/3), & 1/2 \le x \le 1 \end{cases}$$



1. Squish: 2. Stretch: 3. Stack



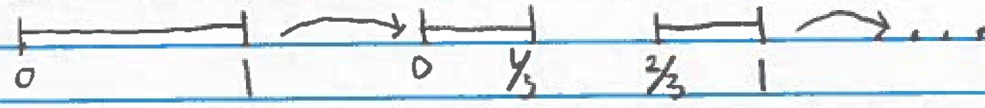




The planar baker's map is chaotic. What is its attracting set?

$$A=\bigcap_{n=0}^{\infty} F^n([0,1]\times[0,1])$$

Cross Sections:



The attracting set is

$$A=[0,1]\times C$$ (C is the Cantor set)

Box dimension is

$$1+\frac{\ln(2)}{\ln(3)}.$$