Lecture 17: Lorenz Equations

3-D Dynamics

Old Tools:

1. Local linearization
2. global analysis

* Trajectories can be attracted to lower dimensional subsets

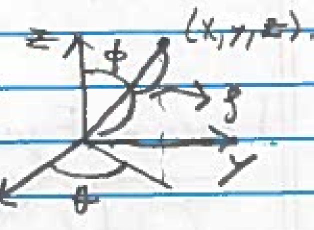
Example:

$$\dot{x} = -y + \frac{x(1-x^2-y^2-z^2)}{\sqrt{x^2+y^2+z^2}}$$

$$\dot{y} = x + \frac{y(1-x^2-y^2-z^2)}{\sqrt{x^2+y^2+z^2}}$$

$$\dot{z} = \frac{z(1-x^2-y^2-z^2)}{\sqrt{x^2+y^2+z^2}}$$

Convert to spherical polar:

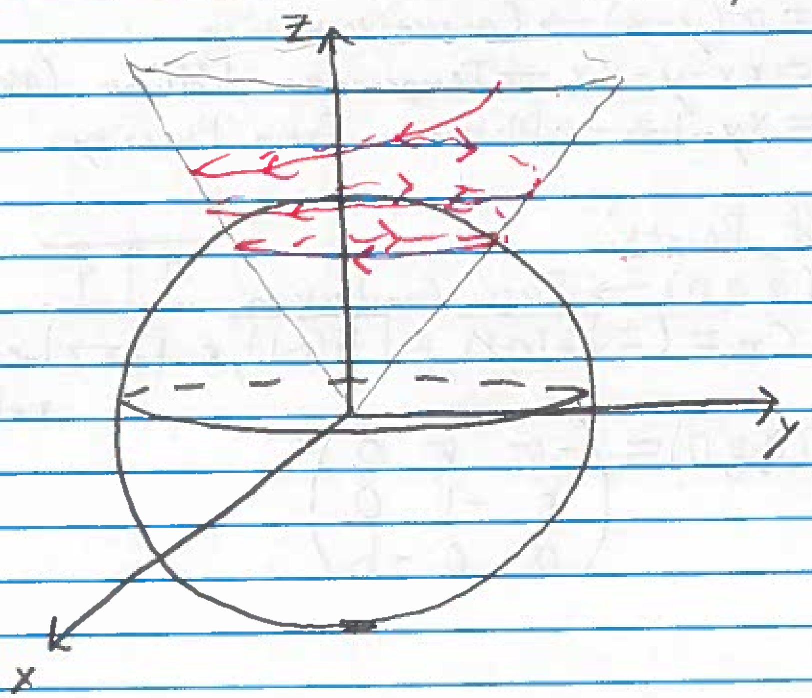


$$\dot{\rho} = 1-\rho^2$$

$$\dot{\theta} = 1$$

$$\dot{\phi} = 0$$

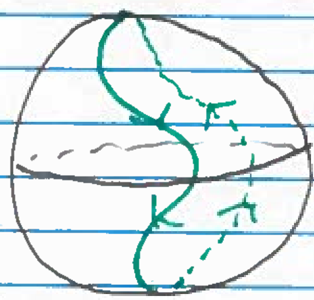
*All trajectories go to sphere of radius 1 and for limit cycles of constant ϕ. That is they approach along a cone.



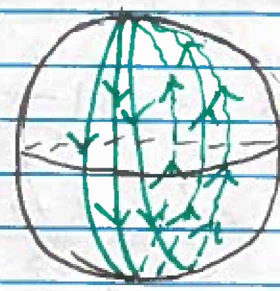
Example:

$$\dot{\rho} = 1-\rho^2$$
$$\dot{\theta} = \theta_p$$
$$\dot{\phi} = \phi_p$$

The dynamics forms Lissajous cycles on the sphere. If $\theta_p/\phi_p \in \mathbb{Q}$ then the curves on the sphere close up. Otherwise they repeat forever. This is known as quasiperiodicity

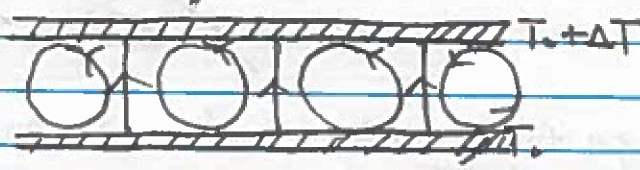


Lissajous



Quasiperiodic Orbit.

Lorenz Equations



$\dot{x} = \sigma(y-x) \rightarrow$ Convective motion
$\dot{y} = rx-y-xz \rightarrow$ Temperature difference (ascending/descending)
$\dot{z} = xy-bz \rightarrow$ Distortion from linearity.

Fixed Points:

1. $(0,0,0) \rightarrow$ Pure Conduction
2. $C_\pm = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1) \rightarrow$ Left and right moving rolls.

$$J(0,0,0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

Case 1:

$r<1$, only one fixed point and it is a stable node.

* We can show there are no closed orbits by constructing a Lyapunov function.

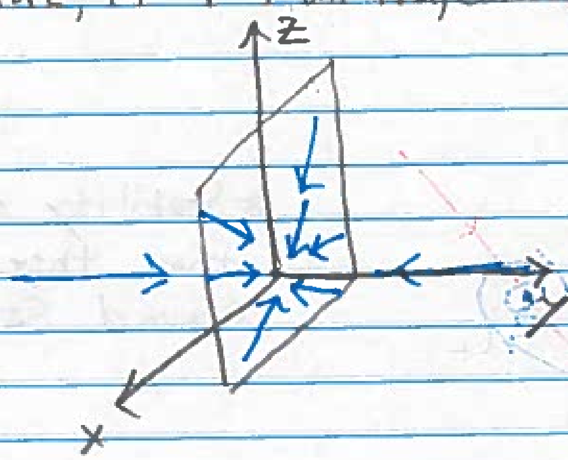
Let $V = \frac{1}{2}\left(\frac{1}{\sigma}x^2 + y^2 + z^2\right)$

$$\Rightarrow \dot{V} = \frac{1}{\sigma}x\dot{x} + y\dot{y} + z\dot{z}$$
$$= x(y-x) + y(rx-y-xz) + z(xy-bz)$$
$$= (r+1)xy - x^2 - y^2 - bz^2$$
$$= -\left(x^2 - (r+1)xy + \left(\tfrac{r+1}{2}\right)^2 y^2\right) + \left[\left(\tfrac{r+1}{2}\right)^2 - 1\right]y^2 - bz^2$$
$$= -\left(x - \left(\tfrac{r+1}{2}\right)y\right)^2 - \left(1 - \left(\tfrac{r+1}{2}\right)^2\right)y^2 - bz^2$$

If $r<1$, $\dot{V}<0$ and $V \geq 0$ with $V=0$ if and only if $x=y=z=0$.

$$\Rightarrow \lim_{t\to\infty} V(t) = 0.$$

Therefore, if $r<1$ all trajectories go to the origin.



Case 2:

$1<r<r_s$

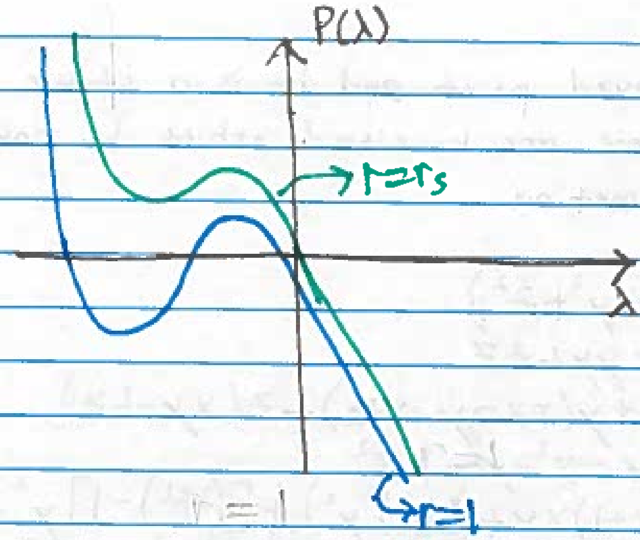
$(0,0,0)$ has eigenvalues $\lambda_1>0$, $\lambda_2<0$, $\lambda_3<0$.

$\Rightarrow (0,0,0)$ is a saddle.

$C_\pm$ now exist $\Rightarrow$ pitch fork bifurcation at $r=1$.

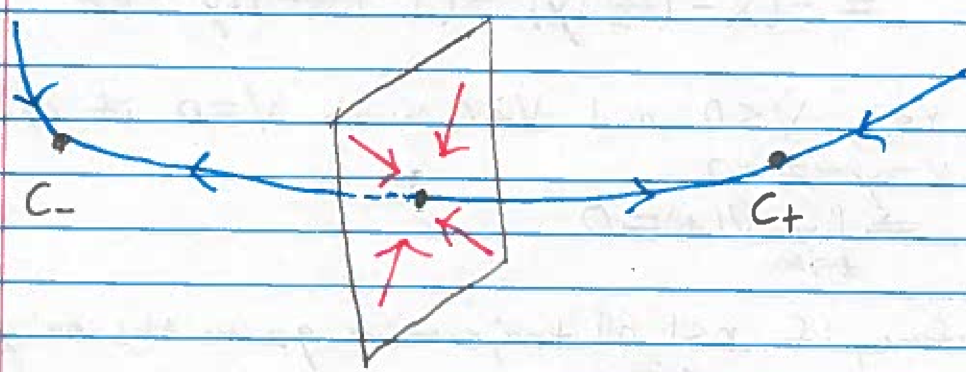
From now on we fix $b = 8/3$ and $\sigma = 10$. The characteristic polynomial of the Jacobian at $C_\pm$ is given by

$$3P(\lambda) = -\lambda^3 - 41\lambda^2 - 88\lambda - 80(r-1).$$



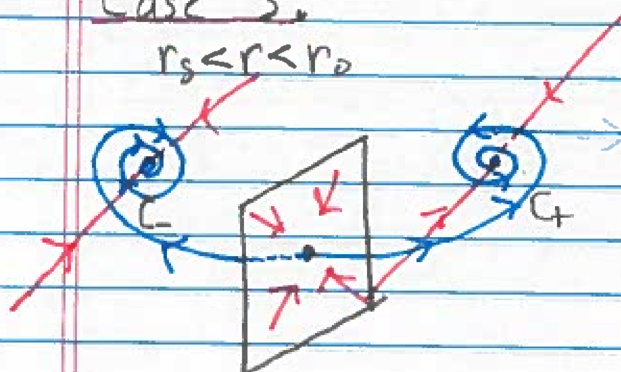
$\Rightarrow$ Changing r shifts this graph

$\Rightarrow$ There exists r_s such that if $1<r<r_s$ then $C_\pm$ are stable.



Case 3:

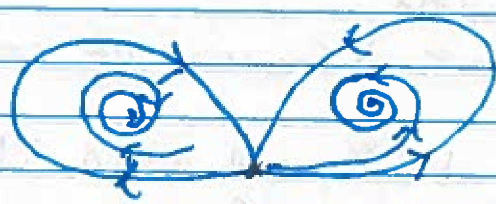
$r_s<r<r_0$



*Stability analysis tells us that they are stable spirals around $C_\pm$.

Case 4:

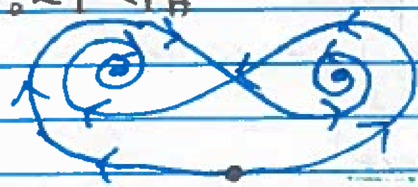
$r=r_0$



*unstable limit cycle born in a homoclinic bifurcation.

Case 5:

$r_0 < r < r_H$



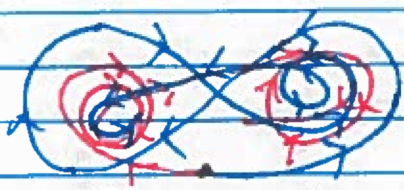
$\rightarrow$ This creates a strange invariant set.

*Spirals may loop around a lot. Impossible to predict which fixed point we go to.

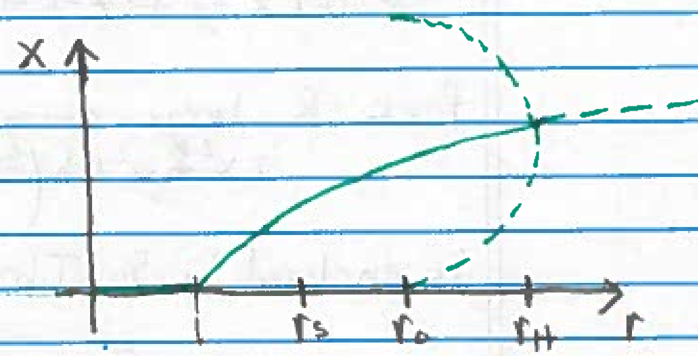
This is called transient chaos.

Case 6:

A Hopf bifurcation occurs at some critical r_H. The unstable limit cycle vanishes leaving behind two unstable spirals.



Strange Attractor



Volume Contraction:

$\dot{\vec{x}} = F(\vec{x})$. Let V_0 denote an initial volume of initial conditions.

$$\Rightarrow \underbrace{\dot{V}}_{\text{rate of change of volume}} = \underbrace{\int_{\partial S} \vec{F} \cdot \vec{n} \, dA}_{\text{flux through the boundary.}}$$

$$\Rightarrow \dot{V} = \int_V \nabla \cdot \vec{F} \, dV$$

$$= \int_V (-\sigma - 1 - b) \, dV$$

$$= (-\sigma - 1 - b) V$$

$$\Rightarrow V(t) = V_0 \exp(-(\sigma + 1 + b)t)$$

All volumes shrink to nothing!!

$\Rightarrow$ The trajectories must be attracted to something...

1. Can't be a surface.
2. Can't be a fixed point.
3. Possible to eliminate limit cycles.
4. Can't be quasiperiodic.

$\Rightarrow$ Must be a new object!

Trajectories Remain Bounded

Consider a spherical surface:

$$S_R:\ x^2+y^2+(z-R-\sigma)^2=R^2$$

For a trajectory starting on the surface we have that

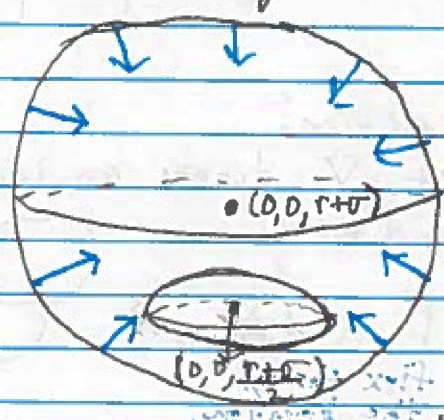
$$\frac{d}{dt}\Big[\underbrace{x^2+y^2+(z-R-\sigma)^2}_{\text{square distance from } (0,0,R+\sigma)}\Big]=2x\dot{x}+2y\dot{y}+2(z-r-\sigma)\dot{z}$$

$$\Rightarrow \frac{d}{dt}\left[x^2+y^2+(z-R-\sigma)^2\right]=-2\left[\sigma x^2+y^2+b\left(z-\frac{r+\sigma}{2}\right)^2-\frac{b(r+\sigma)^2}{4}\right]$$

Pick R large enough so that the ellipse

$$\sigma x^2+y^2+b\left(z-\frac{r+\sigma}{2}\right)^2=\frac{b(r+\sigma)^2}{4}$$

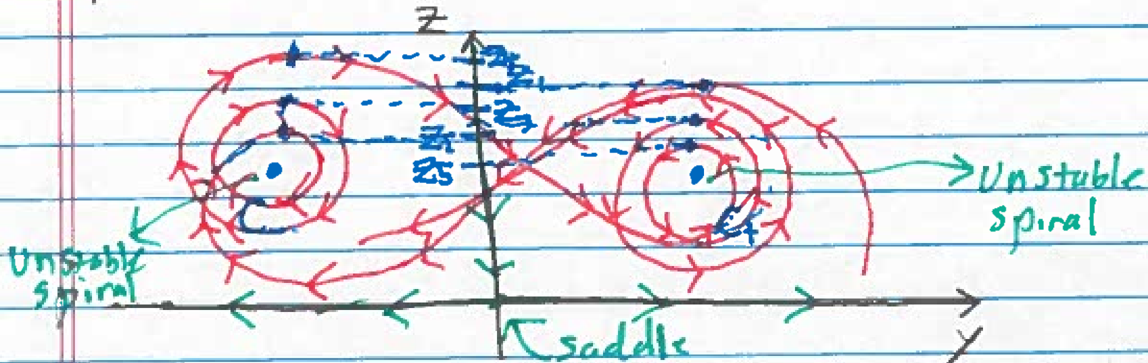
is enclosed in S_R. This guarantees S_R is a trapping region.



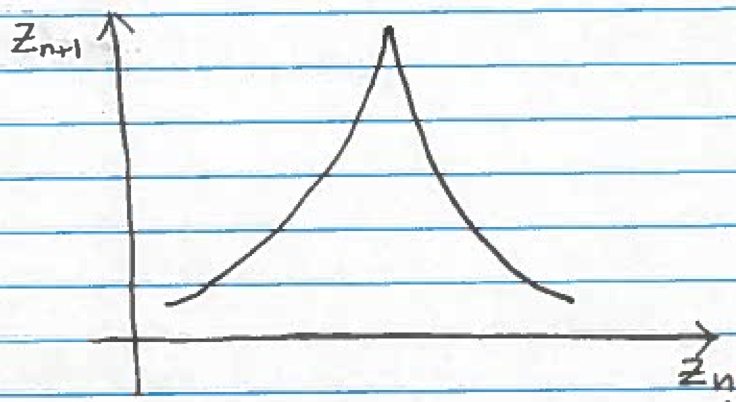
$\Rightarrow$ Trajectories remain bounded and must go to something.

Lorenz Map

Take a particular trajectory and project onto y-z plane. Let z_n denote local maximums of the trajectory.



The Lorenz map look at the relationship between z_n and z_{n+1}



This is essentially the tent map!!

Consequence:

1. The dynamics is chaotic on the strange attractor.
2. There is an infinite number of unstable limit cycles.
3. The saddle acts to stretch the dynamics. The unstable spirals fold the trajectories.