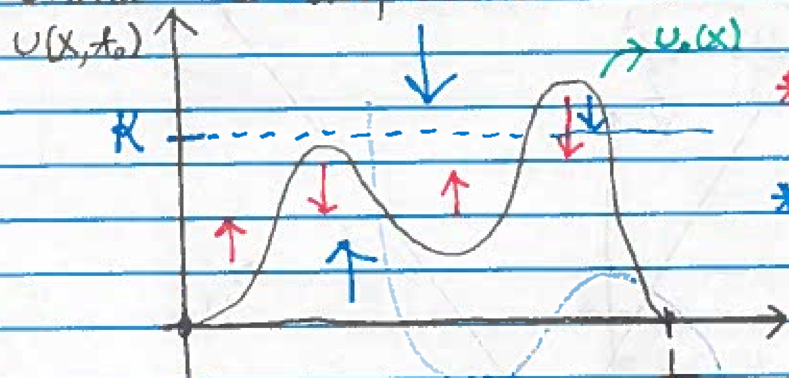
Lecture 18: Reaction Diffusion Equations

Logistic Equation:

Imagine a species (bacteria) in a 1-d petri dish. Let

$u(x,t) =$ population density as a function of space x and time t.

Consider a snapshot at time t_0:



* Population in peaked regions spreads out

* Population below carrying capacity rises.

Model:

$$u_t = \alpha u_{xx} + ru(1 - u/K)$$ — Competition between pop. growth and diffusion

$$u(0,t) = 0$$
$$u(L,t) = 0$$

zero population at edge of petri dish.

$$u(x,t_0) = u_0(x)$$ — Initial Distribution of population.

Rescale:

$$X = x/L,$$
$$\tau = rt$$
$$U = u/K$$

$$\Rightarrow \frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial X} \Rightarrow \frac{\partial^2}{\partial x^2} = \frac{1}{L^2}\frac{\partial^2}{\partial X^2}$$

$$\frac{\partial}{\partial t} = r\frac{\partial}{\partial \tau}$$

The Rescaled Equation:

$$U_\tau = DU_{XX} + U(1-U)$$
$$U(0,\tau) = 0$$
$$U(1,\tau) = 0$$
$$U(X,\tau_0) = U_0(X).$$

IGNORE.

Steady State Solutions:

The analogs of fixed points satisfy:

$$0 = u^*_{xx} + u^*(1-u^*)$$

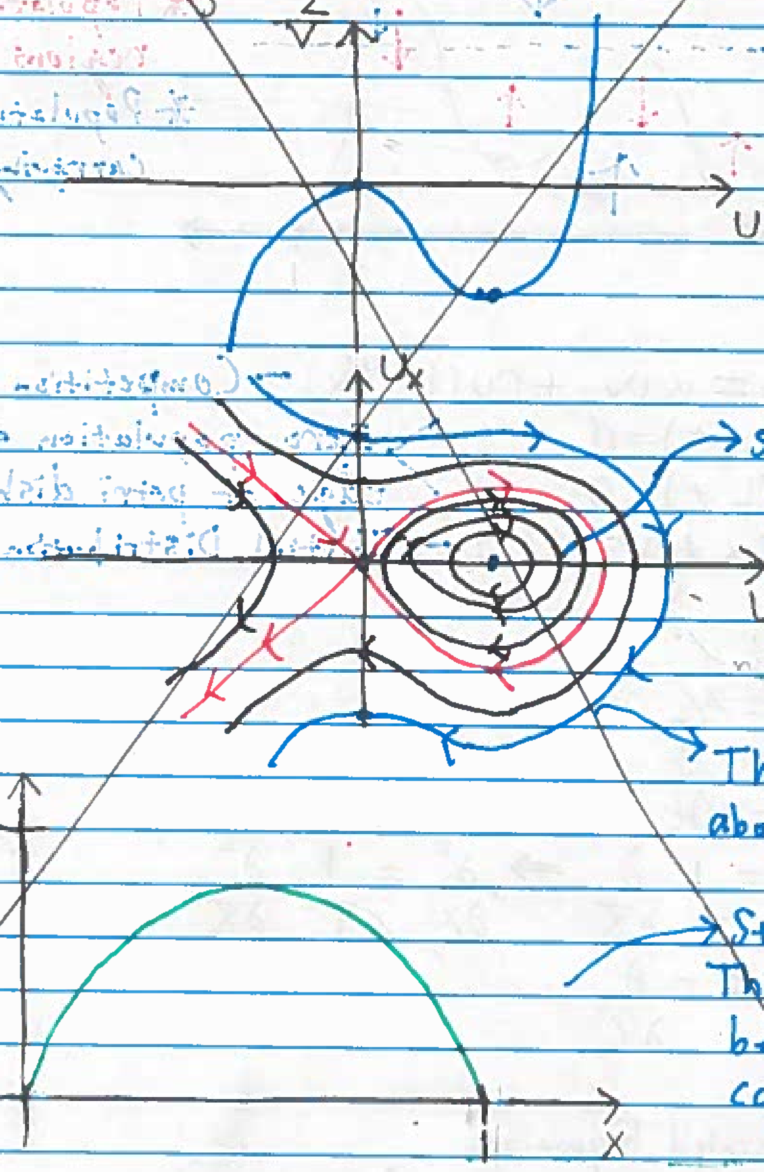
$$\Longrightarrow u^*_{xx} = -u^*(1-u^*) \quad *$$

This is now an ordinary differential equation.

We can use phase portraits to find stationary solutions.

$*$ is a conservative system with potential

$$V = \frac{u^3}{3} - \frac{u^2}{2}$$



Solution that satisfies $u(0,t)=0$, $u(L,t)=0$

This trajectory goes above the carrying capacity.

Steady state solution. The solution is always below the carrying capacity.

Steady State Solutions

The analogs of fixed points satisfy:

$$0 = D u^*_{xx} + u^*(1-u^*)$$

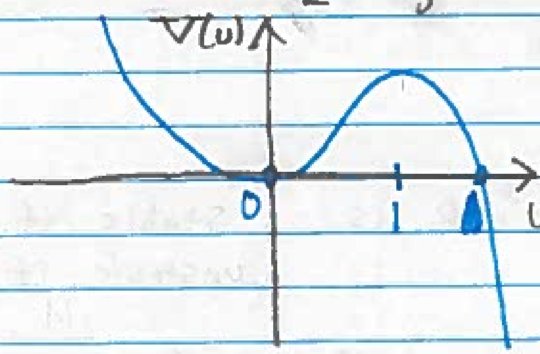
$$\Rightarrow u^*_{xx} = -u^*(1-u^*). \quad (*)$$

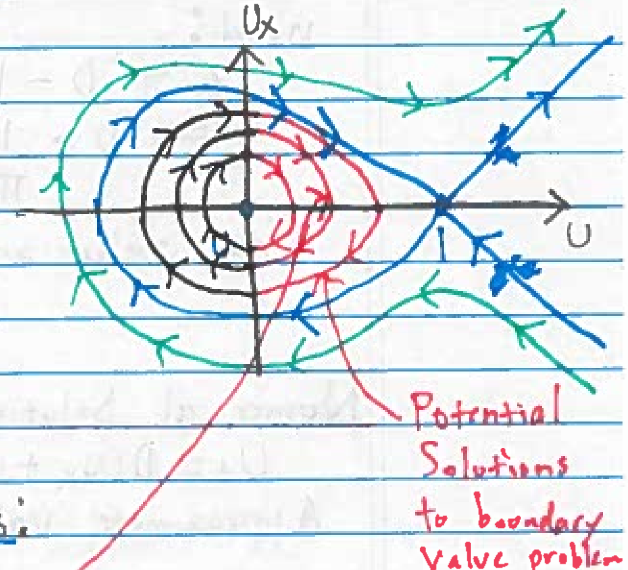
*This is now an ordinary differential equation

*We can use phase portraits to find stationary solutions.

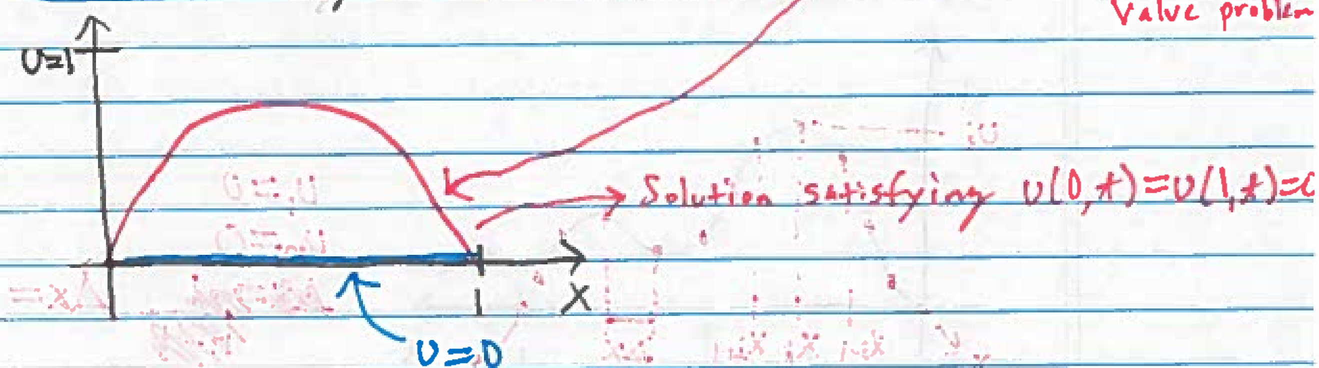
$(*)$ is a conservative system with potential

$$V(u) = \frac{u^2}{2} - \frac{u^3}{3}.$$





Potential Steady State Solutions:



Stability?

How do small perturbations change in time??

*Let $u = 0 + \delta u$ and assume $u_t = D u_{xx} + u(1-u)$

$$\Rightarrow \delta u_t = D\,\delta u_{xx} + \delta u(1-\delta u)$$

Assume δu is small and drop higher order terms:

$$\delta u_t = D\,\delta u_{xx} + \delta u$$

Make an ansatz of the form:

$$\delta u = A e^{i(\lambda t + \nu x)}$$

$$\Rightarrow i\lambda = -\nu^2 D + 1$$

$$\Rightarrow \lambda = i(\nu^2 D - 1)$$

Generically solutions are of the form:

$$\delta u = e^{-(\omega^2 D - 1)t}\left(A\cos(\omega x) + B\sin(\omega x)\right)$$

$$\delta u(0,t) = A e^{-(\omega^2 D-1)t} = 0$$

$$\Rightarrow A = 0$$

$$\delta u(1,t) = e^{-(\omega^2 D-1)t} B\sin(\omega)$$

$$\Rightarrow \omega = n\pi,\quad n \geq 1,$$

$$\Rightarrow \delta u(x,t) = e^{-(n^2\pi^2 D - 1)t} B\sin(n\pi x)$$

To ensure the perturbation decays for all $n \in \mathbb{N}$ we need:

$$\pi^2 D - 1 > 0$$

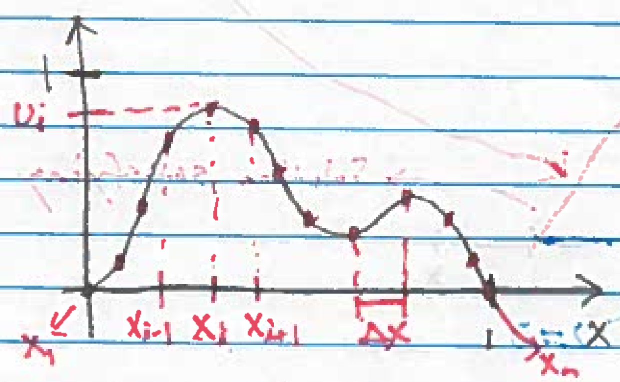
$$\Rightarrow D > \frac{1}{\pi^2}.$$

The solution $u^* \equiv 0$ is $\begin{cases} \text{stable} & \text{if } D > \frac{1}{\pi^2} \\ \text{unstable} & \text{if } D < \frac{1}{\pi^2} \end{cases}$.

Numerical Solution Logistic Equation:

$$U_t = D U_{xx} + U(1-U)$$

Approximate value of U at discrete locations.



$U_1 = 0$

$U_n = 0$

$\Delta x = \dfrac{1}{n-1}$

- $\dfrac{\partial u_i}{\partial x} \approx \dfrac{U_i(x_{i+1}) - U(x_i)}{\Delta x} = \dfrac{U_{i+1} - U_i}{\Delta x}$ (forward difference)
- $\dfrac{\partial u_i}{\partial x} \approx \dfrac{U(x_i) - U(x_{i-1})}{\Delta x} = \dfrac{U_i - U_{i-1}}{\Delta x}$ (backward difference)
- $\dfrac{\partial u_i}{\partial x} \approx \dfrac{U_{i+1} - U_{i-1}}{2\Delta x}$ = average of forward difference and backward difference. (*)

$$\approx \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}$$

How do we approximate second derivative?

$$\frac{\partial^2 u_i}{\partial x^2} \approx \frac{u_x(x+\Delta x/2) - u_x(x-\Delta x/2)}{2\,\Delta x/2} \quad \text{By } (*)$$

$$\approx \frac{\dfrac{u(x+\Delta x)-u(x)}{2\,\Delta x/2} - \dfrac{u(x)-u(x-\Delta x)}{2\,\Delta x/2}}{\Delta x} \quad \text{By } (*) \text{ again.}$$

$$\Longrightarrow \frac{\partial^2 u_i}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}.$$

We now approximate Logistic equation by a system of ordinary differential equations:

$$\begin{cases} \dot{u}_1 = 0 \\ \dot{u}_2 = \dfrac{D}{\Delta x^2}(u_3 - 2u_2 + u_1) + u_2(u_2 - 1) \\ \dot{u}_3 = \dfrac{D}{\Delta x^2}(u_4 - 2u_3 + u_2) + u_3(u_3 - 1) \\ \vdots \\ \dot{u}_{n-1} = \dfrac{D}{\Delta x^2}(u_n - 2u_{n-1} + u_{n-2}) + u_{n-1}(u_{n-1} - 1) \\ \dot{u}_n = 0 \end{cases}$$

Let
$$A = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \rightarrow \text{Row } n-2 \\ \rightarrow \text{Row } n-1 \\ \rightarrow \text{Row } n \end{array}$$

We can write the above system as:

$$\dot{\vec{u}} = \frac{D}{\Delta x^2} A\vec{u} + \underbrace{\vec{u}(\vec{u}-1)}_{\text{Componentwise multiplication.}}$$

To numerically simulate this equation we now approximate the time derivative

$$\frac{\vec{U}(t+\Delta t)-\vec{U}(t)}{\Delta t} = \frac{D}{\Delta x^2} A\vec{U} + \vec{U}(\vec{U}-1)$$

$$\Rightarrow \vec{U}(t+\Delta t) = \vec{U}(t) + \left(\frac{D}{\Delta x^2} A\vec{U} + \vec{U}(\vec{U}-1)\right)\Delta t$$

For numerical stability we need to satisfy the Courant-Friedrichs-Lewy (CFL) condition:

$$\frac{D\Delta t}{\Delta x^2} < 1$$

$$\Rightarrow \boxed{\Delta t < \frac{\Delta x^2}{D}}$$

*See Matlab code for implementation.

Predator Prey Equation:

$$u_t = Du_{xx} + au - buv$$
$$v_t = Dv_{xx} - cv + duv$$
$$u(x,0) = u_0(x)$$
$$v(x,0) = v_0(x)$$
$$u(0,t) = u(1,t) = 0$$
$$v(0,t) = v(1,t) = 0$$

Steady State Solutions

$$Du_{xx} = -au + buv$$
$$Dv_{xx} = -cv - duv$$

One obvious solution is $u=0, v=0$. We now check is stability by considering

$$u = 0 + \delta u, \quad v = 0 + \delta v$$

$$\Rightarrow \delta u_t = D\delta u_{xx} + a\delta u - b\delta u\delta v$$
$$\delta v_t = D\delta v_{xx} - c\delta v + d\delta u\delta v$$

Assume $\delta u, \delta v$ are small and drop higher order terms:

$$\Rightarrow \delta u_t = D\delta u_{xx} + a\delta u$$
$$\delta v_t = D\delta v_{xx} - c\delta v$$

We make an ansatz of the form:

$$\delta u = A e^{i(\lambda_1 t + \nu_1 x)}, \quad \delta v = B e^{i(\lambda_2 t + \nu_2 x)}$$

$$\Rightarrow i\lambda_1 = -D\nu_1^2 + a$$
$$i\lambda_2 = -D\nu_2^2 - b$$

$$\Rightarrow \lambda_1 = i(D\nu_1^2 - a)$$
$$\lambda_2 = i(D\nu_2^2 + b)$$

$$\Rightarrow \delta u(x,t) = e^{-(n^2\pi^2 D - a)t}\sin(n\pi x)$$
$$\delta v(x,t) = e^{-(n^2\pi^2 D - b)t}\sin(n\pi x).$$

Therefore, $u=0$, $v=0$ is stable if $D > a/\pi^2$.