Lecture 2: Flows on the Line

General Framework:

$\dot{x} = f(x)$, $x \in \mathbb{R}$ $f: \mathbb{R} \to \mathbb{R}$ differentiable

x - position

$f(x)$ - velocity (really f is a vector field)

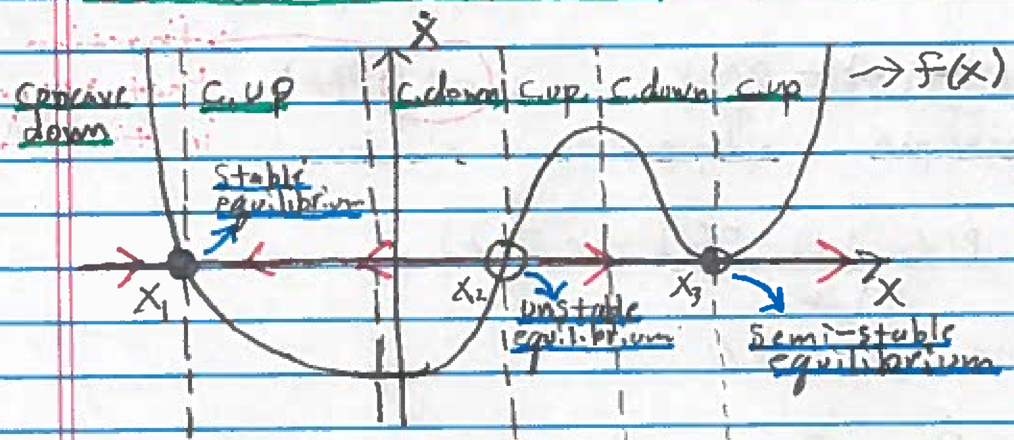
We can try to solve:

$$t = \int_{x_0}^{x} \frac{1}{f(s)}\, ds$$

i) We may not be able to integrate explicitly

ii) It may not be possible to solve for $x(t)$.

Geometric Point of View:



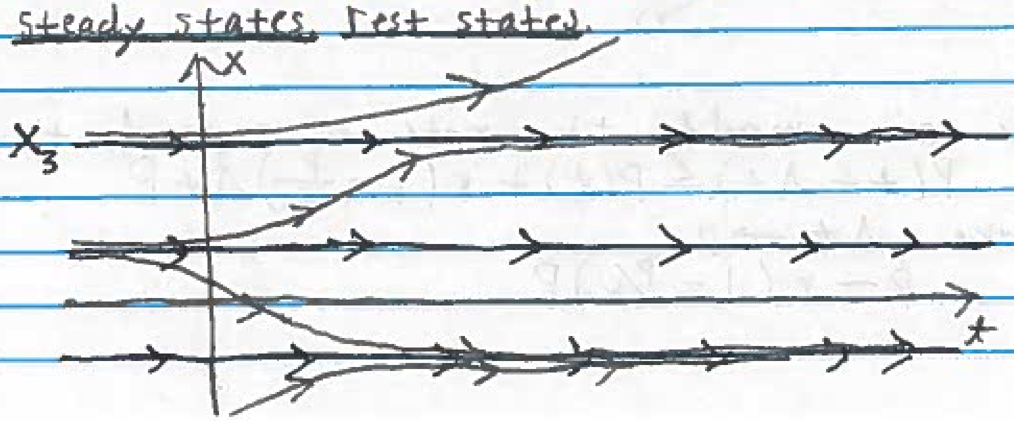
$$\ddot{x} = \frac{d}{dt}\frac{dx}{dt} = \frac{d}{dt} f(x(t)) = \frac{df}{dx}\frac{dx}{dt} = \frac{df}{dx} \cdot f$$

i) $f(x) > 0$, $\frac{dx}{dt} > 0 \Rightarrow x(t)$ increases, particle moves right

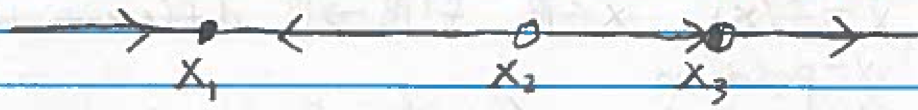
ii) $f(x) < 0$, $\frac{dx}{dt} < 0 \Rightarrow x(t)$ decreases, particle moves left

iii) $f(x) = 0$, $\frac{dx}{dt} = 0$, $x(t) = x_0$ is a solution of $\frac{dx}{dt} = f(x)$

Solutions to $f(x) = 0$ are called fixed points, equilibrium, steady states, rest states.



The phase portrait captures all of this behavior



At each point in $\mathbb{R}$ the function f assigns a tangent vector pointing left or right. Local stability can be determined graphically.

Example (Population Growth):
Infinite number of resources, no predators.

$$\underbrace{P(t+\Delta t)}_{\text{new pop.}} = \underbrace{P(t)}_{\text{old population}} + \underbrace{r\Delta t\, P(t)}_{\text{children}}$$

$r\Delta t$ → fraction that reproduce in time Δt.

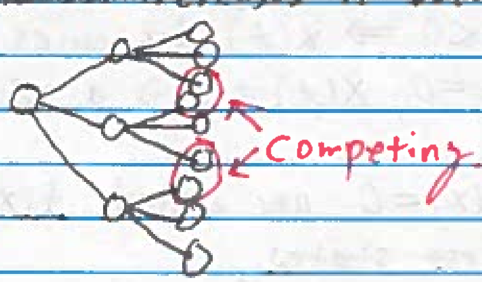
$$\Rightarrow \frac{P(t+\Delta t)-P(t)}{\Delta t} = r P(t)$$

Take $\Delta t \to 0$

$$\Rightarrow \dot{P} = rP$$

$$\Rightarrow P = P_0 \exp(rt)$$

A more realistic model must account for finite resources.
As population increases it becomes harder to reproduce

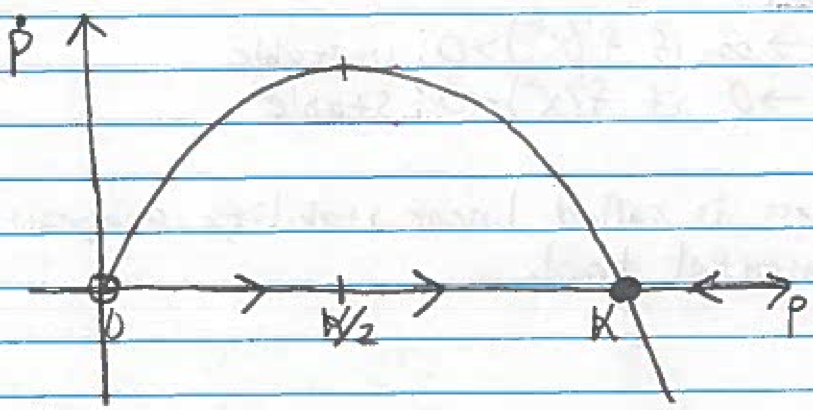


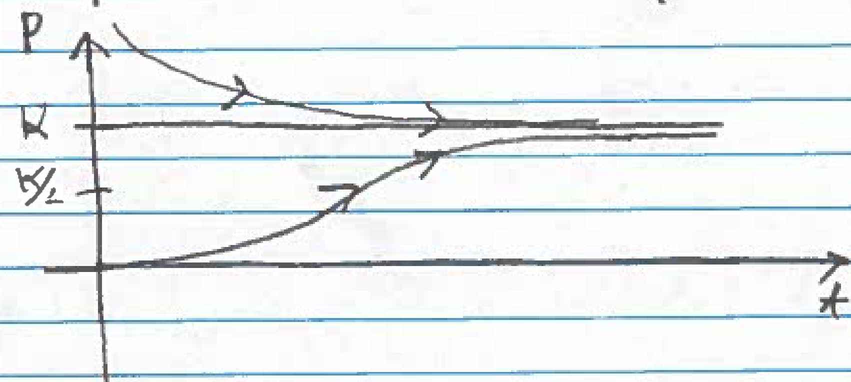
We can modify the rate of reproduction:

$$P(t+\Delta t) = P(t) + r\left(1-\frac{P}{K}\right)\Delta t\, P$$

Take $\Delta t \to 0$

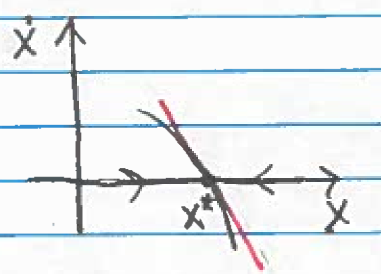
$$\dot{P} = r(1-P/K)P$$



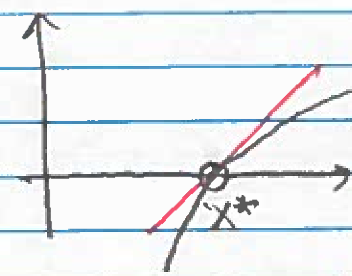


Population goes to carrying capacity.

Analytical Approach:



Stable, $f'(x^*)<0$



unstable, $f'(x^*)>0$.

Taylor expand near x^*:

$$f(x) = \underbrace{f(x^*)}_{=0} + f'(x^*)(x-x^*) + \ldots$$

Let $y = x - x^*$ and admit higher order terms:

$$\dot{y} \approx f'(x^*)\,y$$

$$\rightarrow y \approx y(0)\exp(f'(x^*)\,t)$$

Conclusion:

1. $y(t) \to \infty$ if $f'(x^*) > 0$; unstable
2. $y(t) \to 0$ if $f'(x^*) < 0$; stable

This process is called linear stability analysis and is a fundamental tool.