Lecture 3: Saddle Node Bifurcations

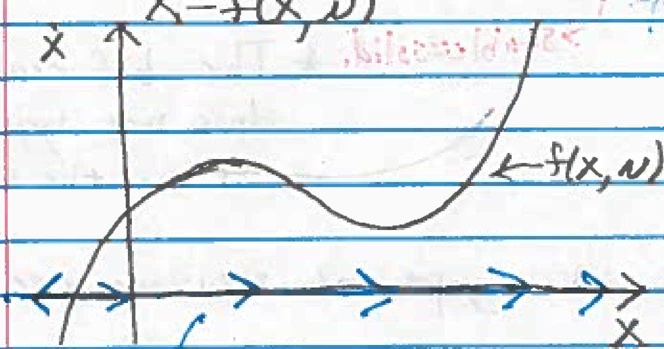
Systems with parameters can have drastic qualitative changes as a parameter is varied.

Examples:

* Euler Buckling, * Turbulence, * Outbreaks of epidemics.
* Catastrophic environmental collapse.

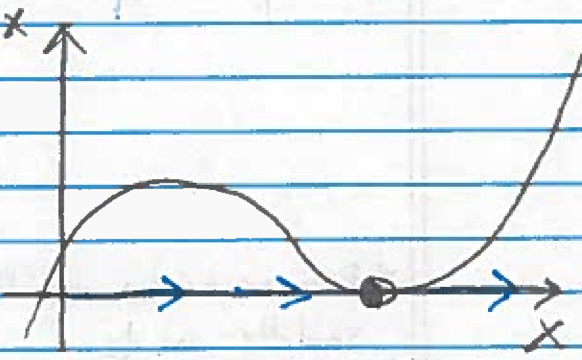
Example (Graphical):

$\dot{x} = f(x, \mu)$



$\mu < \mu^*$

All flow is ~~to the right~~ unstable.



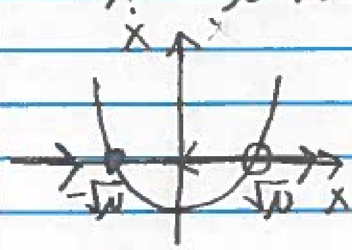
$\mu = \mu^*$

$\mu > \mu^*$

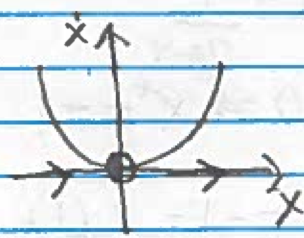
The bifurcation point is ~~the~~ the value of μ where the fixed point changes stability.

Example (Normal Form):

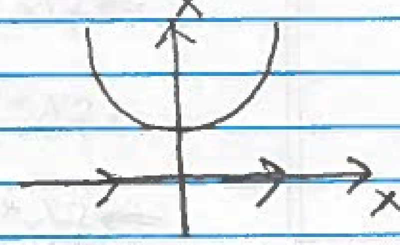
$\dot{x} = -\mu + x^2$



$\mu > 0$

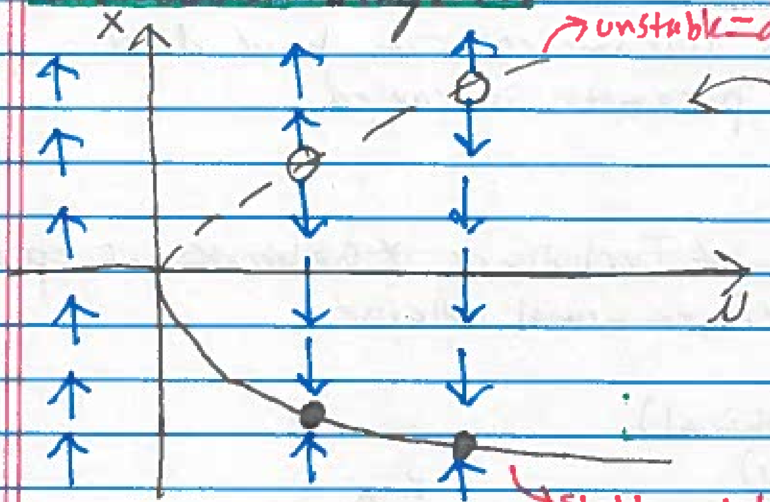


$\mu = 0$



$\mu < 0$

Bifurcation Diagram:



* location of fixed points as a function of u.

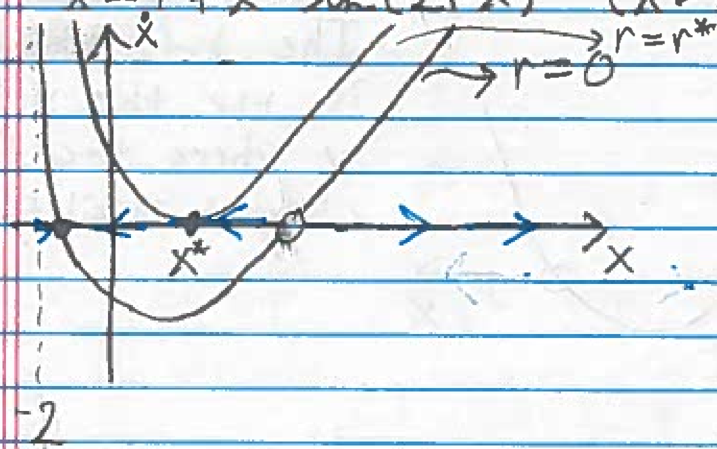
* The phase portraits for a fixed u can be reconstructed

* The bifurcation diagram does not typically contain the arrows.

* Bifurcations with this type of structure are called saddle node

Example:

$\dot{x} = r + x - \ln(2+x) \quad (x > -2)$



* At bifurcation point $f(x^*) = 0$ and $f'(x^*) = 0$

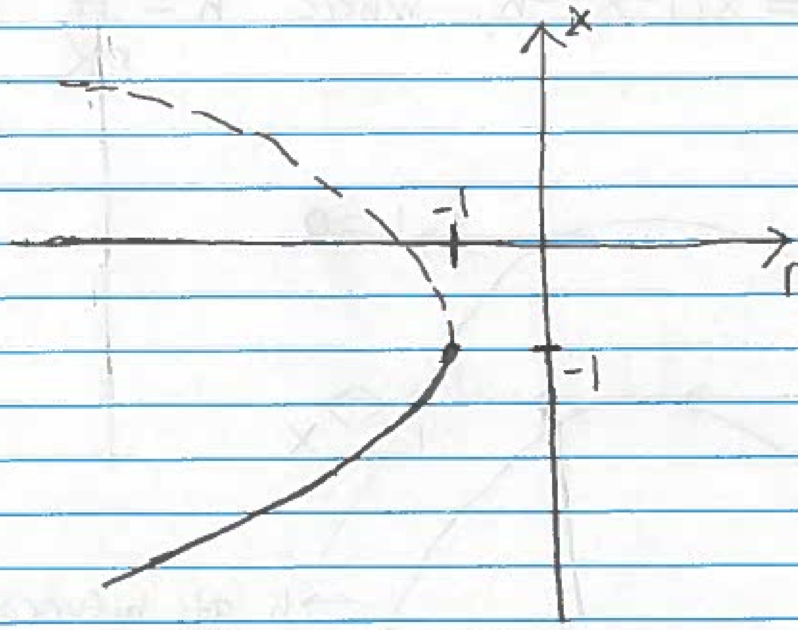
Let $f(x) = r + x - \ln(2+x)$

$\Rightarrow f'(x) = 1 - \dfrac{1}{2+x}$

$\Rightarrow f'(x^*) = 0 \Rightarrow x^* = -1$

$\Rightarrow f(x^*) = r - 1 - \ln(1)$

$\Rightarrow r^* = -1$



Bifurcation Diagram.

Example:

Model of fishing in a lake

$$\dot{N} = \underbrace{rN(1-N/K)}_{\text{logistic growth}} - \underbrace{H}_{\text{harvesting.}}$$

N - population of fish
r - rate of growth
K - carrying capacity in absence of harvesting
H - harvesting (set by # of fishing liscences).

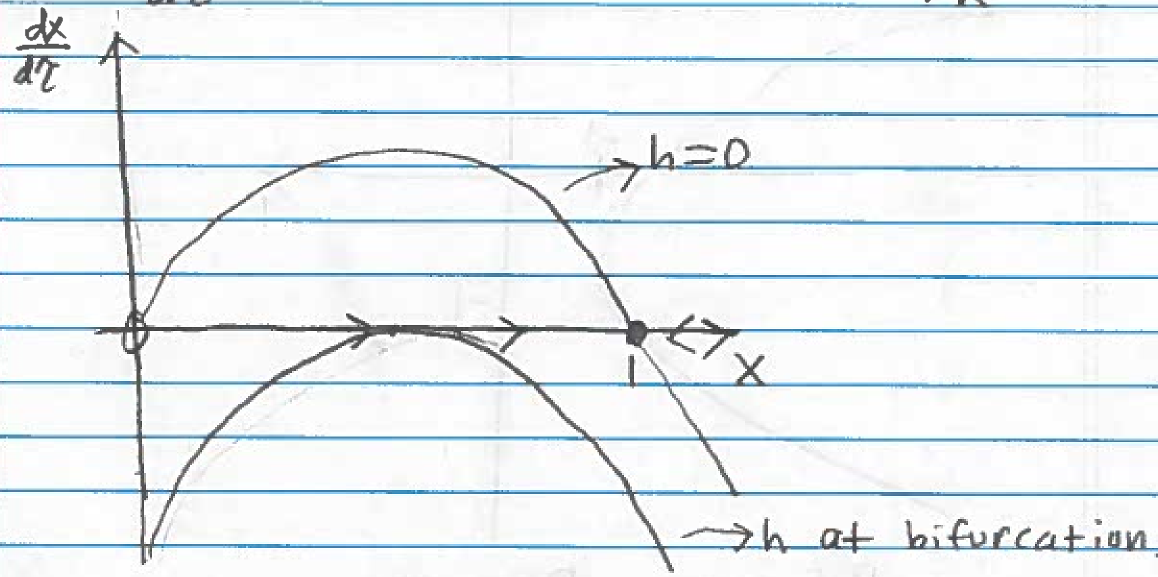
Let $\tau = rt$, $x = N/K$ → dimensionless scales

$$\Rightarrow \frac{dN}{dt} = \frac{dN}{d\tau}\frac{d\tau}{dt} = r\frac{dN}{d\tau}$$

$$\Rightarrow r\frac{dN}{d\tau} = rN(1-N/K) - H$$

$$\Rightarrow rK\frac{dx}{d\tau} = rKx(1-x) - H$$

$$\Rightarrow \frac{dx}{d\tau} = x(1-x) - h, \quad \text{where} \quad h = \frac{H}{rK}.$$



Let $f(x) = x(1-x) - h$

$f'(x) = 1 - 2x = 0$

$\Rightarrow x = 1/2$

Now, $f(1/2) = 1/4 - h \Rightarrow h = 1/4$ is the bifurcation point.

If $h > 1/4$ the fish all die. $\Rightarrow H > rK/4$

* This model allows $h < 0$, we will later do a fixed version of this model.

* A key step was introducing the variables $\tau = rt$ and $x = N/K$, We will learn how to do this in a systematic manner.