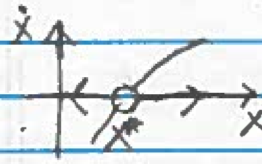
Lecture 4: Transcritical and Pitchfork Bifurcations

Framework:

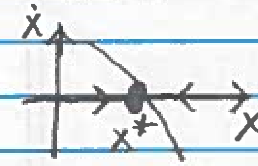
$$\dot{x} = f(x,\nu),\quad x\in\mathbb{R},\ \nu\in\mathbb{R}$$

Recall:

1. $\left.\dfrac{\partial f}{\partial x}\right|_{(x^*(\nu),\nu)} > 0 \Rightarrow$ unstable fixed point



2. $\left.\dfrac{\partial f}{\partial x}\right|_{(x^*(\nu),\nu)} < 0 \Rightarrow$ stable fixed point



3. $\left.\dfrac{\partial f}{\partial x}\right|_{(x^*(\nu),\nu)} = 0 \Rightarrow$ possibly semistable

We can think of x^* as a (multivalued) function of ν.

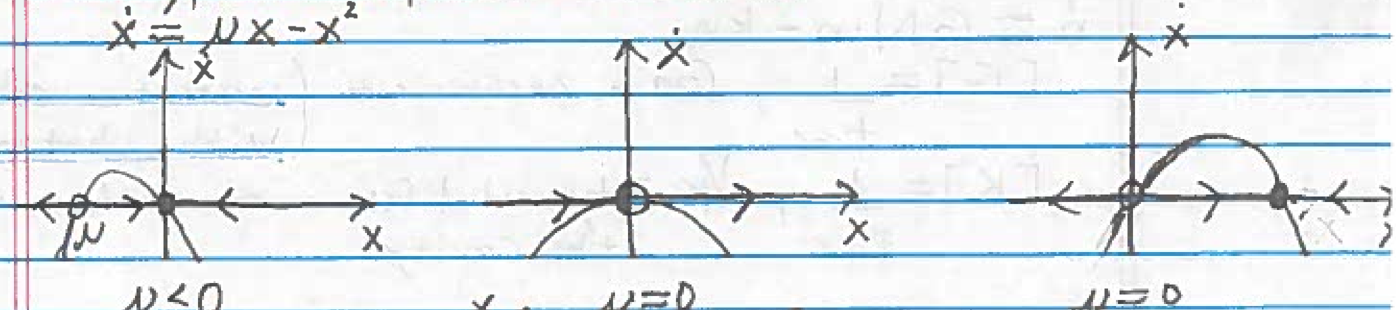
* Analytically, bifurcations occur when $\left.\frac{\partial f}{\partial x}\right|_{(x^*(\nu),\nu)} = 0$.

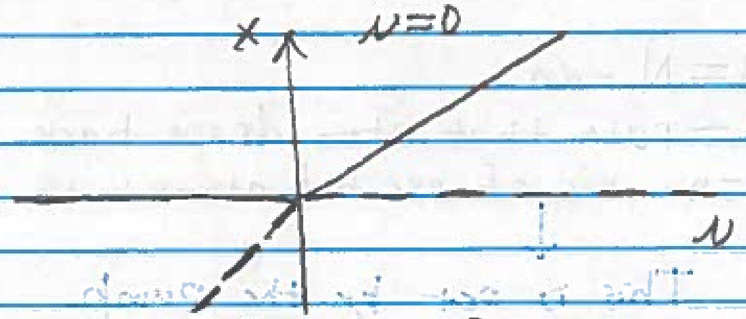
Transcritical Bifurcation:

Many systems have a fixed point that cannot change position (population $P=0$, for example). However, the stability of the fixed point can change.

Prototypical Example (Normal Form):

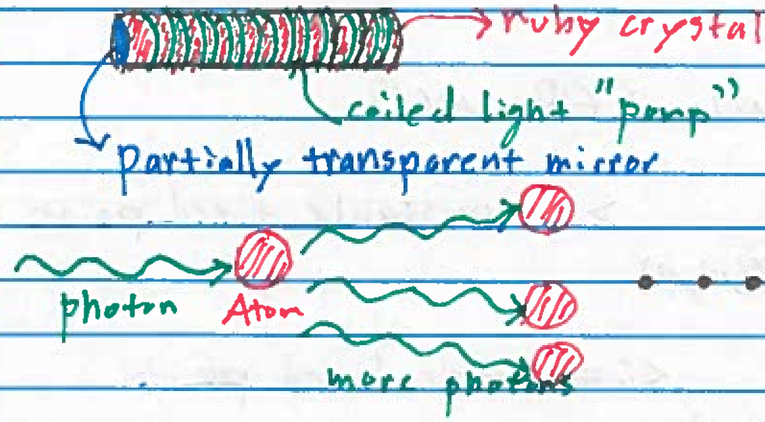
$$\dot{x} = \nu x - x^2$$





Bifurcation Diagram.

Example (Model of a laser):



n – number of photons
N – number of excited atoms

1. Photon interacts with atom raising its energy. This is called "exciting" the atom

2. Excited atoms spontaneously drop to lower energy state emitting more photons.

3. By constantly pumping in light we hope to start a chain reaction in which n becomes constant (lasing).

$$\dot{n} = GN\cdot n - kn$$

$[G] = \frac{1}{\text{time}}$, Gain coefficient (interaction of atoms with photons)

$[K] = \frac{1}{\text{time}}$, $1/K$ = typical lifetime of photon inside the cavity.

$$N(t) = N_0 - \alpha n$$

α – rate that atom drops back to rest state.

N_0 – number of excited atoms with no photons

This is set by the pump.

$$\Rightarrow \dot{n} = G(N_0 - \alpha n)n - kn$$
$$= n \cdot (GN_0 - \alpha G n - k)$$

Fixed Points:

$$n = 0, \quad n = \frac{GN_0 - k}{G\alpha}$$

Let $f(n) = (GN_0 - k)n - G\alpha n^2$

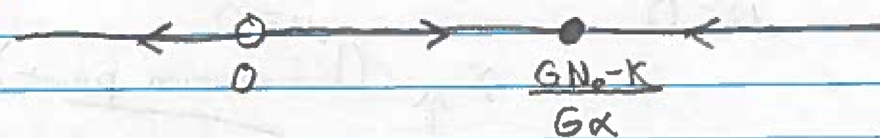
$$\Rightarrow f'(n) = GN_0 - k - 2G\alpha n$$

Test fixed points:

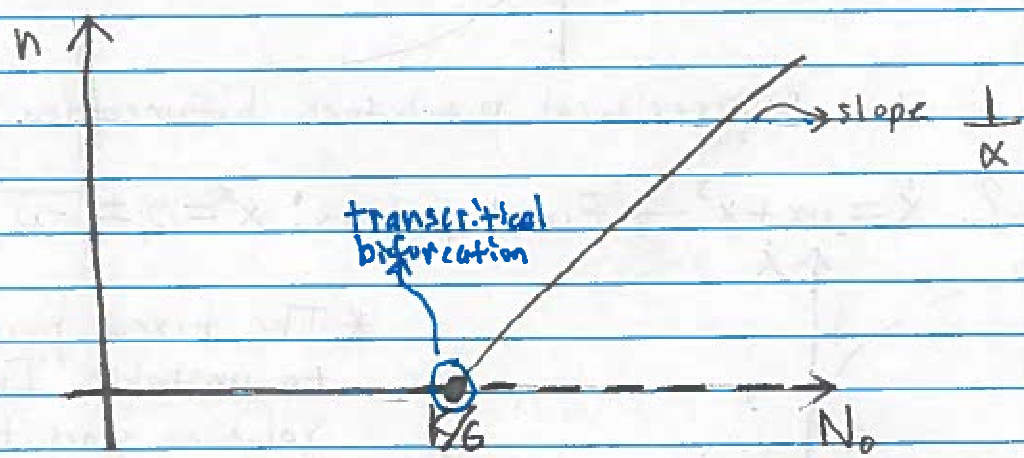
$$f'(0) = GN_0 - k$$

0 is stable if and only if $N_0 < k/G$

* Graphically if 0 is unstable then $(GN_0 - k)/G\alpha$ is stable



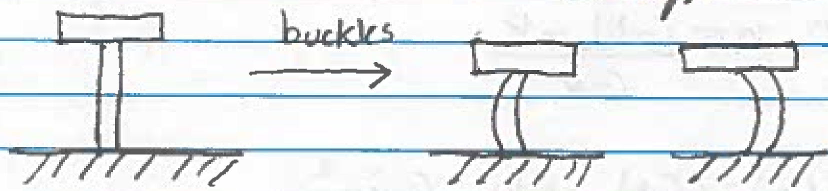
This is a nice trick! For a saddle node bifurcation a graphical analysis was easier while for a transcritical bifurcation the analytical approach is easier.



Pitchfork Bifurcations

Many physical systems have odd symmetry:

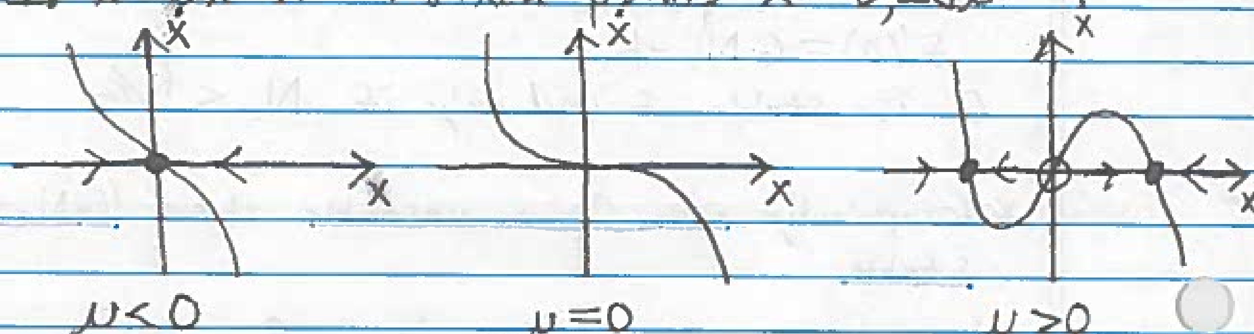
$f(-x,\mu) = -f(x,\mu)$ (evenly symmetric potential energy)

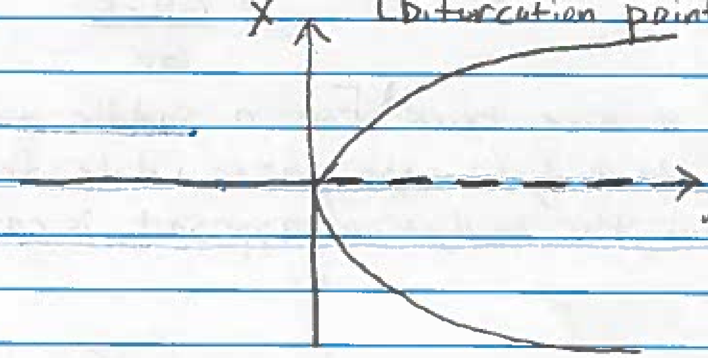


(buckling left or right is equally likely)

Example (Normal Forms):

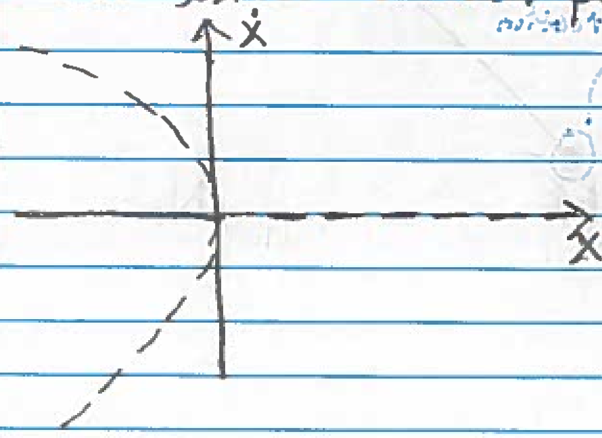
1. $\dot{x} = \mu x - x^3 \Rightarrow$ fixed points $x^* = 0, \pm\sqrt{\mu}$





supercritical pitchfork bifurcation diagram

2. $\dot{x} = \mu x + x^3 \rightarrow$ fixed points: $x^* = 0, \pm\sqrt{-\mu}$



* The fixed point switches to unstable. In fact, the solution goes to ∞ in finite time.