Lecture 5: Improved Model of a Lake

Old Model:

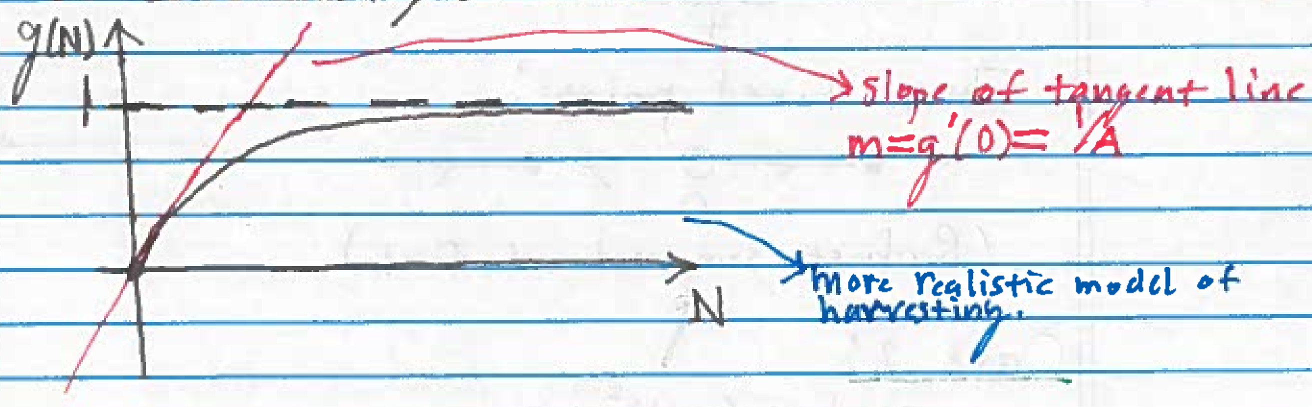
$$\dot{N} = rN\left(1-\frac{N}{K}\right) - H$$

* N can become negative.
* If N is small, it should be harder to harvest fish
* If N is large, fish can be caught as fast the fisherman fish, i.e. a constant rate.
* If $N=0$, harvesting $=0$.

New Model:

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - H\cdot\frac{N}{A+N} = \underbrace{rN\left(1-\frac{N}{K}\right)}_{\text{logistic growth}} - H\underbrace{g(N)}_{\text{fraction actually caught.}}$$

r – pop. growth rate
K – carrying capacity
H – harvesting rate
A – survivability of fish



Let $x = N/K$, $\tau = rt$

$$\Rightarrow \dot{x} = x(1-x) - \frac{hx}{a+x}$$

Fixed Points:

$$x^* = 0, \quad x^* = \frac{(1-a) \pm \sqrt{(1-a)^2 + 4(a-h)}}{2}$$

Fix a, what happens as a changes?

*Look at stability at $x=0$. Let $f(x)=x(1-x)-\frac{hx}{a+x}$.

$$\Rightarrow f'(x)=1-2x-\frac{[(a+x)\cdot h-hx]}{(a+x)^2}$$

$$\Rightarrow f'(0)=1-ah/a^2=(a-h)/a.$$

$\Rightarrow 0$ is stable if $h>a$, unstable if $h<a$.

*Other roots exist if

$$(a-1)^2+4(a-h)>0$$

$$\Rightarrow h<\frac{(a-1)^2}{4}+a.$$

Case 1:

$$h<a<\frac{(a-1)^2}{4}+a$$

Three fixed points:

(Robust survival of fish).

Case 2:

$$a<h<\frac{(a-1)^2}{4}+a$$

Three fixed points

(Species Endangered: Small changes in addition to harvesting could cause extinction)

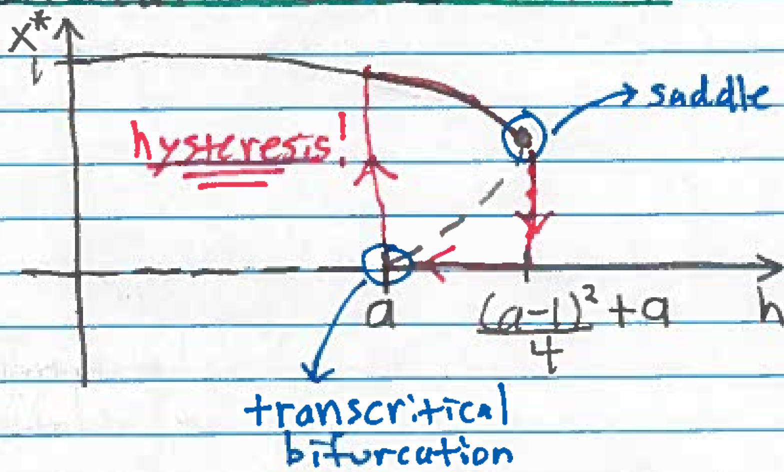
Case 3:

$\frac{(a-1)^2}{4} + a < h$

Only one fixed point

→ extinction of fish.

Bifurcation Curve (Fixed a):



* Hysteresis: If $h > \frac{(a-1)^2}{4} + a$ the fish population suddenly goes extinct. This is an example of a tipping point. To recover the fish population h must be drastically decreased below a.

Phase Diagram: