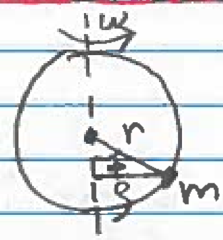
Lecture 6: Bead on a Rotating Hoop



How large must ω be to guarantee stability of mass?

$$\underbrace{mr\ddot{\phi}}_{\text{net force}} = \underbrace{-b\dot{\phi}}_{\text{friction}} \underbrace{-mg\sin(\phi)}_{\text{gravity}} \underbrace{+mr\omega^2\sin(\phi)\cos(\phi)}_{\text{centrifugal}} = 0$$

Dimensional Analysis:

$[m] = M$

$[r] = L$

$[b] = T^{-1}LM$

$[g] = LT^{-2}$

$[\omega] = T^{-1}$

Rescale:

$\tau = T_{sc}\, t$

We will choose T_{sc} later.

$$\Rightarrow \frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = T_{sc}\frac{d}{d\tau}$$

$$\Longrightarrow mr\,T_{sc}^2\frac{d^2\phi}{d\tau^2} = -b\,T_{sc}\frac{d\phi}{d\tau} - mg\sin(\phi) + mr\omega^2\sin(\phi)\cos(\phi) = 0$$

Divide by mg to nondimensionalize:

$$\frac{r}{g}T_{sc}^2\frac{d^2\phi}{d\tau^2} = \frac{-b}{mg}T_{sc}\frac{d\phi}{d\tau} - \sin(\phi) + \frac{r\omega^2}{g}\sin(\phi)\cos(\phi) = 0$$

In order to reduce to a first order differential equation we need

$$\varepsilon = \frac{r}{g}T_{sc}^2 \ll 1, \quad \frac{b}{mg}T_{sc} = 1$$

$$\Rightarrow \boxed{T_{sc} = \frac{mg}{b} \Rightarrow \varepsilon = \frac{rm^2g}{b^2} \ll 1}$$

$$\Rightarrow \varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin(\phi) + \gamma \sin(\phi)\cos(\phi)$$

$$\boxed{\gamma = \frac{r\omega^2}{g}}$$

Since $\varepsilon \ll 1$, the 1-D system is then:

$$\frac{d\phi}{d\tau} = \sin(\phi)(\gamma\cos(\phi) - 1)$$

Fixed Points:

$\phi = 0, \pi$, $\cos(\phi) = 1/\gamma$

$$\left.\frac{df}{d\phi}\right|_0 = -1 + \gamma$$

$\Rightarrow$ 1. If $\gamma > 1$, $\phi = 0$ is unstable

2. If $\gamma < 1$, $\phi = 0$ is stable

$$\left.\frac{df}{d\phi}\right|_\pi = 1 + \gamma$$

$\Rightarrow$ $\phi = \pi$ is unstable

$\cos(\phi) = 1/\gamma$