Lecture 7: Modeling the Spread of Diseases

How do ideas/diseases/memes spread?

Assumptions:

1. Population remains constant.
2. There is a susceptible population S.
3. There is an infected population I.
4. When an infected population encounters a susceptible person there is a probability that the susceptible becomes infected.
5. Infected individuals return to susceptible population with some probability.

Model 1:

$$\dot{S} = \alpha I - \beta I S \quad ,$$
$$\dot{I} = \beta I S - \alpha I$$

Since $\dot{S} + \dot{I} = 0$ there exists a constant N (total pop. size) such that $S + I = N$.

$[\alpha] = 1/T$ (recovery rate)
$[\beta] = 1/\text{pop}\cdot T$ (infection rate/population)
$[N] = \text{pop.}$ (total population)

Since $S + I = N$ it follows that:

$$\dot{I} = \beta I (N - I) - \alpha I.$$

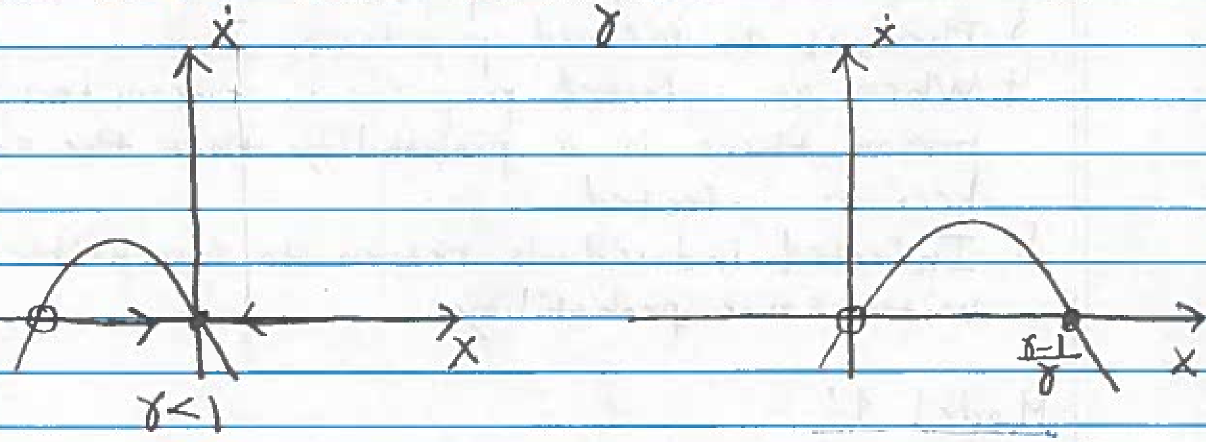
Let $\tau = \alpha t$ and $x = I/N$ then

$$N \alpha \frac{dx}{d\tau} = \beta N^2 x(1-x) - \alpha N x$$
$$\Rightarrow \frac{dx}{d\tau} = x(\gamma(1-x) - 1)$$
$$= -x(\gamma x + 1 - \gamma),$$

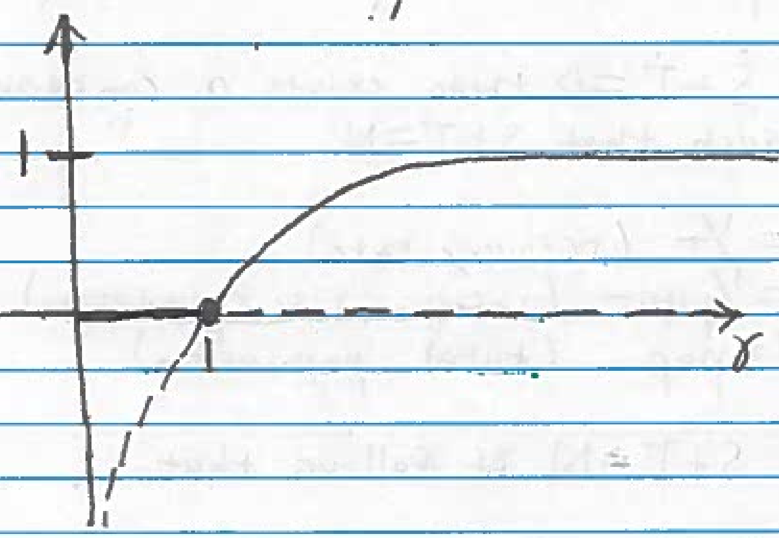
where $\gamma = \frac{\beta N}{\alpha}$ measures the ratio of infection rate to recovery rate.

The fixed points are given by:

$x = 0$ and $x = \frac{\gamma - 1}{\gamma}$.



The bifurcation diagram is:



Consequently, if $\frac{\beta N}{\alpha} > 1$ the disease will reach an endemic state.

Model 2:

What happens if some infected can die?

$$\dot{S} = \alpha I - \beta I S$$
$$\dot{I} = \beta I S - \alpha I - \gamma I \qquad R = \#\text{ of dead people}$$
$$\dot{R} = \gamma I$$

$[\gamma] = 1/T$ is the death rate.

Again $N = S + I + R$ is constant.

$$\Rightarrow S = N - I - R$$

Therefore, the model reduces to:

$$\dot{I} = \beta I (N - I - R) - (\alpha + \gamma) I$$
$$\dot{R} = \gamma I$$

How do we analyze?

Rescale Variables:

$$\tau = (\alpha + \gamma) t$$
$$x = I/N$$
$$y = R/N$$

$$\Rightarrow (\alpha+\gamma) N \frac{dx}{d\tau} = \beta N^2 x (1 - x - y) - (\alpha + \gamma) N x$$
$$(\alpha+\gamma) N \frac{dy}{d\tau} = \gamma N x$$

$$\Rightarrow \boxed{\begin{aligned} \frac{dx}{d\tau} &= \delta x (1 - x - y) - x \\ \frac{dy}{d\tau} &= \rho x \\ \delta = \frac{\beta N}{\alpha+\gamma}, &\quad \rho = \frac{\gamma}{\alpha+\gamma} \end{aligned}}$$

Nullclines:

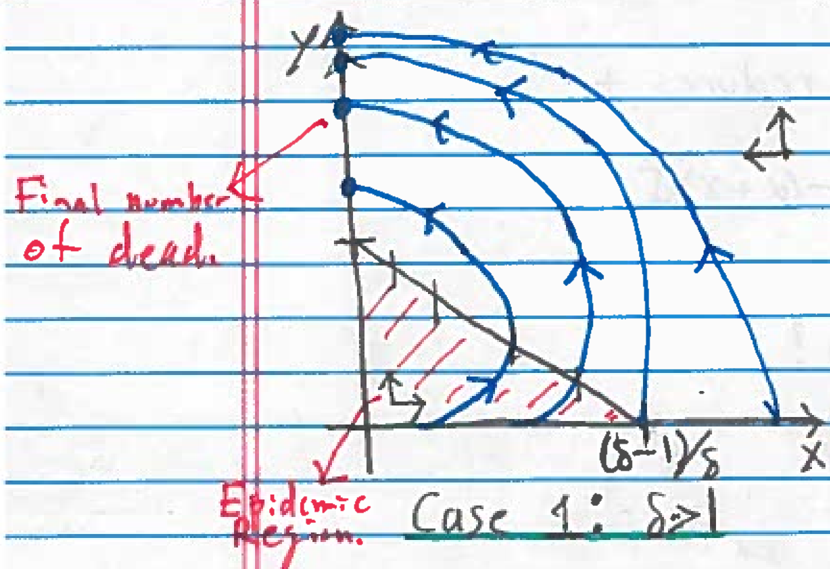
curves along which $\frac{dx}{d\tau}=0$ or $\frac{dy}{d\tau}=0$.

N1: $x=0$ ($\frac{dx}{d\tau}=0$)

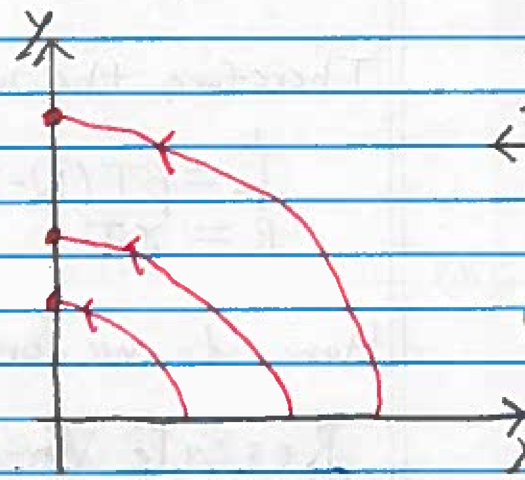
N2: $y=-\delta x+\delta-1$ ($\frac{dx}{d\tau}=0$)

N3: $x=0$ ($\frac{dy}{d\tau}=0$)

Fixed Points where $\frac{dy}{d\tau}=0$ and $\frac{dx}{d\tau}=0$.



Case 1: $\delta>1$



Case 2: $\delta<1$

Consequences:

If $\delta>1$, the infected can increase before the disease burns out. This is an epidemic!

If $\delta<1$, the number of infected is always decreasing.

If $y(0)=0$ and $x(0)$ is small then if $\delta>1$ the disease becomes an epidemic.