Lecture 8: Phase Plane

Big Picture:

Differential Equations with two variables (x,y):

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

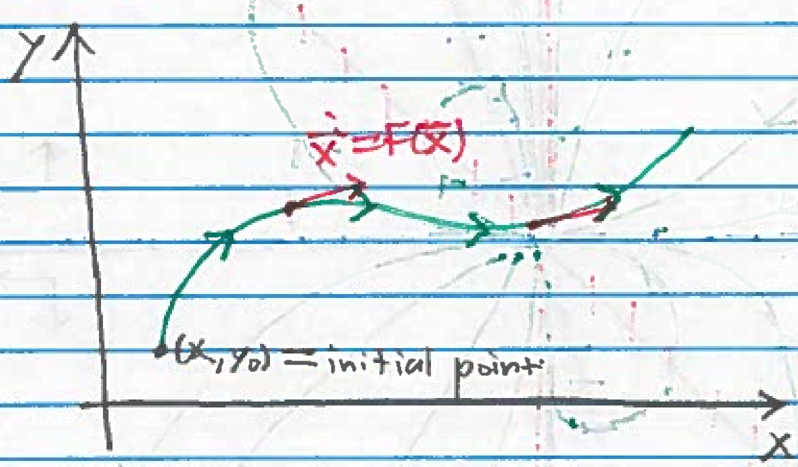
where $(x,y) \in \mathbb{R}^2$ and $f,g: \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable.

We can also write the system in the form:

$$\dot{\vec{x}} = F(\vec{x})$$

where $F: \mathbb{R}^2 \to \mathbb{R}^2$.

Idea: Think of solution curves as flows.



We can think of $F(\vec{x})$ as assigning a velocity vector to each point: to each $\vec{x} = (x,y)$ we assign the vector $F(\vec{x})$.

Solutions of interest:

1. Fixed Points: Each $\vec{x}_0$ satisfying $F(\vec{x}_0) = 0$.
2. Periodic Trajectories: $\vec{x}(t)$ is periodic if there exists $T > 0$ such that $\vec{x}(t+T) = \vec{x}(t)$ for all t and $\vec{x}(t)$ is not a fixed point.

Example:

$\dot{x} = x(x-y)$

$\dot{y} = y(2x-y)$

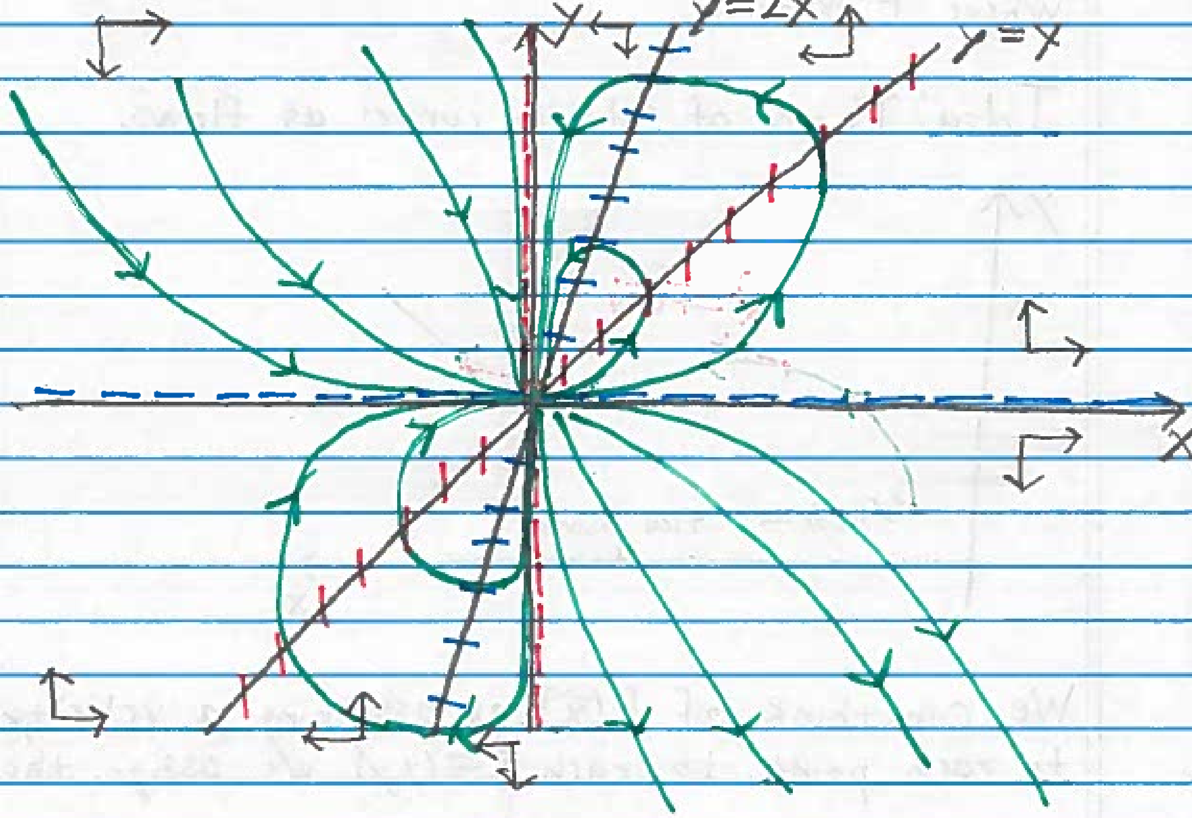
Nullclines are curves along which $\dot{x}=0$ or $\dot{y}=0$.

N1: $x=0$ $(\dot{x}=0)$

N2: $y=x$ $(\dot{x}=0)$

N3: $y=0$ $(\dot{y}=0)$

N4: $y=2x$ $(\dot{y}=0)$



A fixed point $\vec{x}^*$ is stable if there exists $\delta>0$ such that $\|\vec{x}^*-\vec{x}_0\|<\delta$ implies $\lim_{t\to\infty} \vec{x}(t) = x^*$

A fixed point x^* is unstable if it is not stable.