Lecture 9: Linear Systems

Generic System:

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

We can write

$$\dot{\vec{x}} = A\vec{x}$$

with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

1. If $\vec{x}_1, \vec{x}_2$ satisfy $\dot{\vec{x}} = A\vec{x}$ than so does any linear combination:

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

Therefore, it suffices to find two solutions $\vec{x}_1, \vec{x}_2$ with $\vec{x}_1(0)$ and $\vec{x}_2(0)$ linearly independant to generate all solutions.

2. If λ is eigenvalue of A with corresponding eigenvector $\vec{v}$ then $A\vec{v} = \lambda\vec{v}$. Therefore,

$$\vec{x} = e^{\lambda t}\vec{v}$$

is a solution.

proof:

$$\frac{d\vec{x}}{dt} = \lambda e^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v} = A\vec{x}.$$

3. All solutions can be written in the form:

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

where λ_1, λ_2 and $\vec{v}_1, \vec{v}_2$ are the eigenvalues and corresponding eigenvectors, respectively.

4. The eigendirections tell us where the flow is invariant. (Invariant manifolds)

5. At $t=0$ we have

$$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2$$

Example:

1. $\dot{\vec{x}} = \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} \vec{x} = A\vec{x}$

$$\lambda_1 \lambda_2 = \det(A) = -2$$
$$\lambda_1 + \lambda_2 = \mathrm{tr}(A) = -1$$
$$\Rightarrow \lambda_2 = -1 - \lambda_1$$
$$\Rightarrow \lambda_1(1+\lambda_1) = 2$$
$$\Rightarrow \lambda_1^2 + \lambda_1 - 2 = 0$$
$$\Rightarrow \boxed{\begin{matrix} \lambda_1 = -2 \\ \lambda_2 = 1 \end{matrix}}$$

Clearly, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. To determine $\vec{v}_1$:

$$A\vec{v}_1 = -2\vec{v}_1$$
$$\Rightarrow (A + 2I)\vec{v}_1 = 0$$
$$\Rightarrow \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \vec{v}_1 = 0$$
$$\Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The general solution is then:

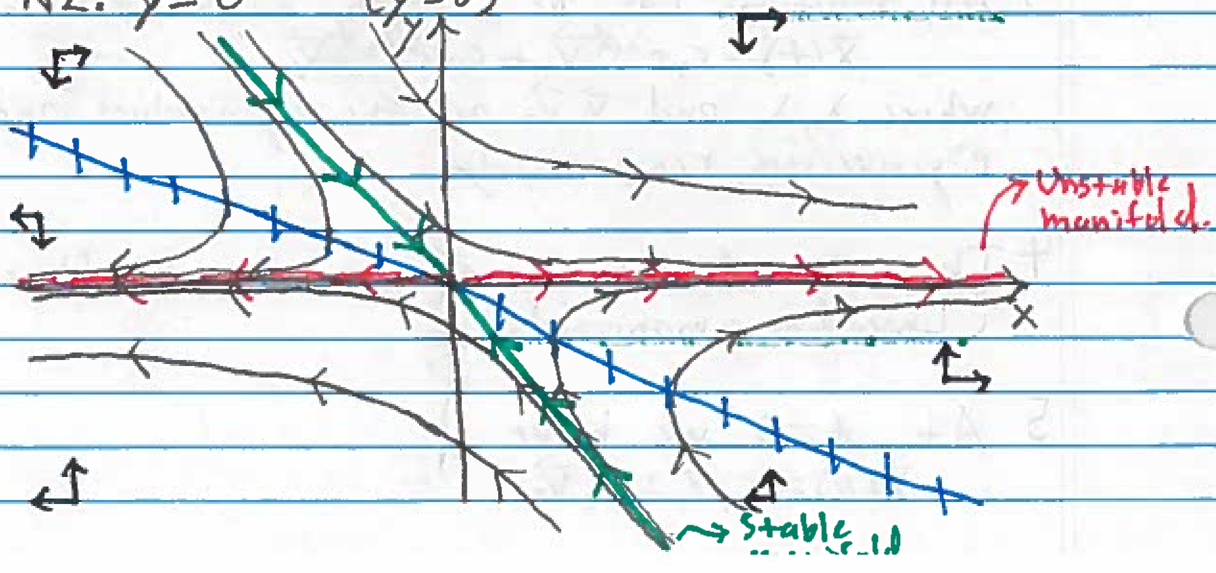
$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Note: $\lim_{t\to\infty} x(t) = \pm\infty$ and $\lim_{t\to\infty} y(t) = 0$.

Phase Portrait:

N1: $y = -\frac{1}{3}x$ $(\dot{x} = 0)$

N2: $y = 0$ $(\dot{y} = 0)$



Example:

$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x} = A\vec{x}$

Eigenvalues satisfy

$\lambda_1 \lambda_2 = 1$

$\lambda_1 + \lambda_2 = 0$

$\Rightarrow \lambda_1 = i, \ \lambda_2 = -i$

The eigenvectors are

$\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

The general solution is:

$$\vec{x}(t) = c_1 e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Recall $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

We want real valued solutions so we set

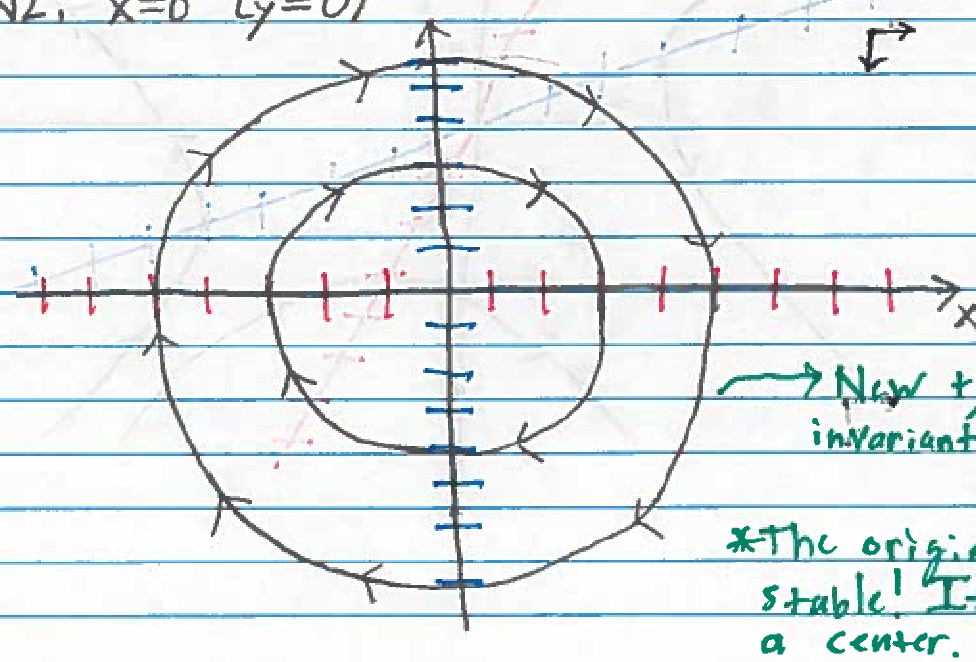
$$\vec{x}(t) = c_1 \operatorname{Re}\left[e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix}\right] + c_2 \operatorname{Im}\left[e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix}\right]$$

$$= c_1 \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$

This gives us the following phase portrait:

N1: $y=0$ $(\dot{x}=0)$

N2: $x=0$ $(\dot{y}=0)$



New type of invariant set.

*The origin is not stable! It is called a center.

Example: (Richardson's Arms Race)

$\dot{x} = ax + by$
$\dot{y} = cx + dy$

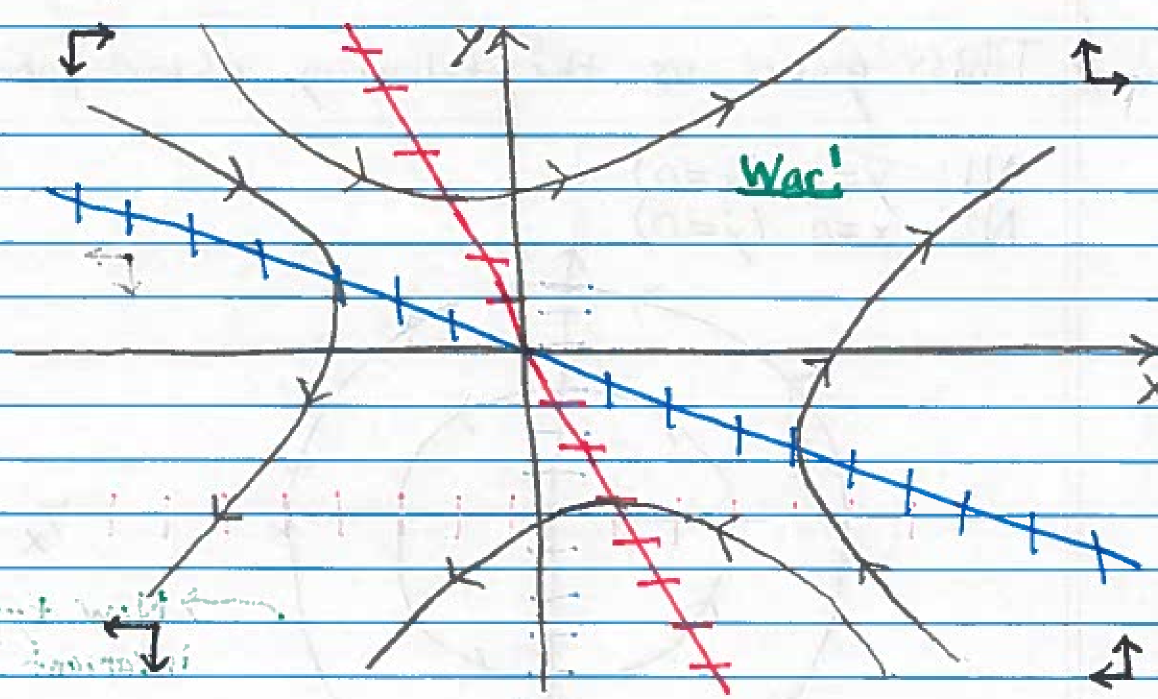
x = expendature on arms by nation x.
y = expendature on arms by nation y.

a,d > 0 nation x or y are arms dealers.
a,d < 0 nation x or y have limited resources.
b,d > 0 nation x or y responds aggresively to other natio
b,d < 0 nation x or y reduces spending depending on the other nations arms.

Case 1:

a,d > 0
b,d > 0

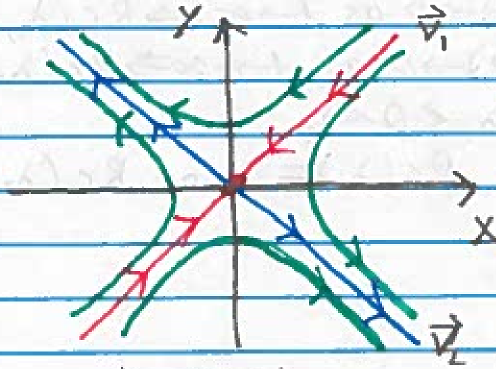
N1: $y = -a/b\,x \quad (\dot{x}=0)$
N2: $y = -c/d\,x \quad (\dot{y}=0)$



Eigenvalue Analysis

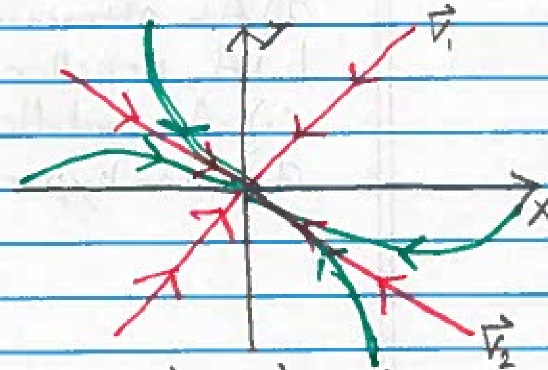
1. $\dot{\vec{x}} = A\vec{x}$, A has two real eigenvalues

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$



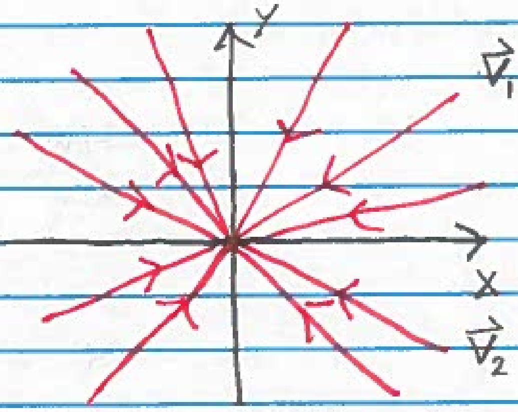
$\lambda_1 < 0 < \lambda_2$

Saddle node



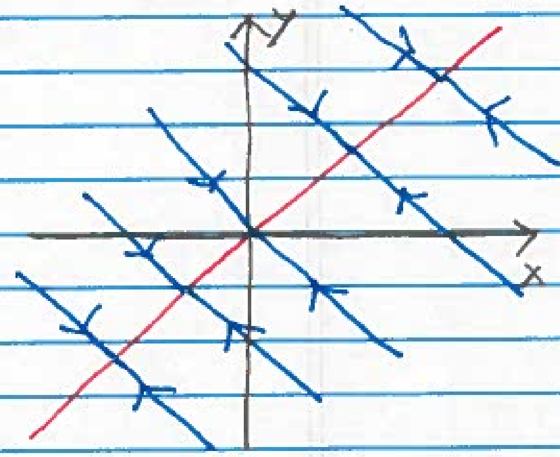
$\lambda_1 < \lambda_2 < 0$

Stable node



$\lambda_1 = \lambda_2 < 0$

Star node



$\lambda_2 < \lambda_1 = 0$

2. A has complex eigenvalues $\lambda, \bar{\lambda} \in \mathbb{C}\setminus\mathbb{R}$ with complex eigenvectors $\vec{v}, \bar{\vec{v}} \in \mathbb{C}^2$

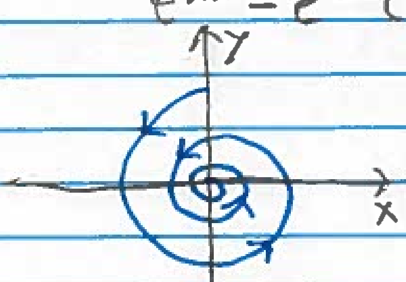
$$\vec{x}(t) = c_1 \,\mathrm{Re}[e^{\lambda t}\vec{v}] + c_2 \,\mathrm{Im}[e^{\lambda t}\vec{v}]$$

If we write

$$\lambda = \nu + i\omega$$

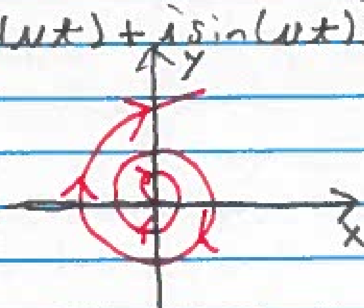
then

$$e^{\lambda t} = e^{\nu t}(\cos(\omega t) + i\sin(\omega t))$$



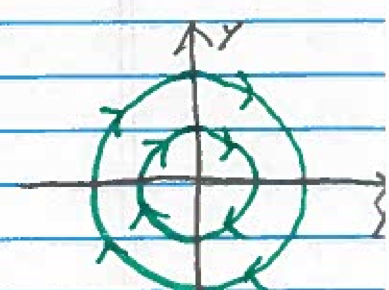
$(\mathrm{Re}(\lambda) < 0)$

Stable Spiral



$(\mathrm{Re}(\lambda) > 0)$

Unstable Spiral



$(\mathrm{Re}(\lambda) = 0)$

Center

Notation:

1. A is called hyperbolic if $Re(\lambda_1), Re(\lambda_2) \neq 0$
2. We say that the fixed point $x=0$ of $\dot{x} = Ax$ is
   - a) An attractor if $\vec{x}(t) \to 0$ as $t \to \infty \Rightarrow Re(\lambda_1, \lambda_2) < 0$
   - b.) A repellor if $\vec{x}(t) \to 0$ as $t \to -\infty \Rightarrow Re(\lambda_1, \lambda_2) > 0$
   - c.) A saddle if $\lambda_1 < 0 < \lambda_2$
   - d.) Nonhyperbolic if $Re(\lambda_1) = 0$ or $Re(\lambda_2) = 0$.