Homework #2

#1

A first-order differential equation is given by
\[ \dot{x} = f(x), \]
where \( f(x) \) is defined by the figure below.

\[ \text{Graph of } f(x) \]

(a) Determine the equilibria.
(b) Determine local stability.
(c) Graph the solution curves \( x(t) \).
(d) What is the limit
\[ \lim_{t \to \infty} x(t) \]
if \( x(0) = 15 \)? What about if \( x(0) = 1 \)?

Solution:
(a) The equilibria are given by
\[ x^* = 0, 5, 8, 11. \]
(b) The phase portrait is given by
\[ \text{Diagram: } 0 \rightarrow 5 \rightarrow 8 \rightarrow 11 \]

Therefore, 0 and 8 are stable while 5 and 11 are unstable.
(d) If \( x(0) = 15 \) it follows that \( \lim_{t \to \infty} x(t) = \infty \).
   If \( x(0) = 1 \) it follows that \( \lim_{t \to \infty} x(t) = 0 \).

#2

The curves \( x(t) \) illustrated below correspond to solution curves for the differential equation \( \dot{x} = f(x) \).

(a) Sketch a one-dimensional phase portrait consistent with this figure.
(b) Sketch a graph of \( f(x) \) consistent with this figure.
(c) Give a formula for \( f(x) \) that is consistent with this figure.
Solutions:
(a) 
(b) 
(c) \( f(x) = (x+1)(x-2)^2 \).

#3.
Consider the following SIS model:
\[
\begin{align*}
\dot{S} &= -\beta S IP S + \alpha I \\
\dot{I} &= \beta IP S - \alpha I \\
\end{align*}
\]
(a) Determine the dimensions of the constants.
(b) Reduce the SIS model to a single differential equation.
(c) Introduce appropriate dimensionless equations.
(d) For the case \( p < 1, q = p - 1 \), determine the threshold condition for the existence of endemic equilibria.
(e) For the case \( p = 1, q = p \), determine the threshold condition for the existence of endemic equilibria.
(f) Why does this model not make sense if \( p \geq 1, p > q \).
Solution:

(a) To balance dimensions it follows that:

$$\begin{align*}
[a] = \frac{1}{\text{time}}, & \quad [\beta] = \frac{1}{\text{time} \cdot \text{pop}^p}, & \quad [\sigma] = \frac{1}{\text{pop}^p}.
\end{align*}$$

(b) Since $$\dot{S} + \dot{T} = 0$$, it follows from conservation of population that:

$$\dot{T} = \beta N^p (N-I) - \alpha T,
\quad 1 + (\sigma N)^{-1}$$

(c) Letting $$x = \frac{T}{N}, \tau = \alpha t$$, it follows that:

$$\begin{align*}
N_0 \frac{dx}{d\tau} &= \beta N^p (1-x)x^p - \alpha N x, \\
\Rightarrow \quad \frac{dx}{d\tau} &= \frac{\beta N^p x(1-x)x^p}{1 + (\sigma N)^{-1}}, \\
\Rightarrow \quad dx &= \frac{A(1-x)x^p - x}{1 + B x^p},
\end{align*}$$

where $$A = \beta N^p / \alpha, \quad B = (\sigma N)^{-1}$$. 

(d) Solving for the fixed point it follows that:

$$\begin{align*}
A(1-x)x^p - x &= 0, \\
1 + B x^p &= 0.
\end{align*}$$

$$\Rightarrow A(1-x)x^p - x(1+Bx^p) = 0 \quad (\ast)$$

$$\Rightarrow x^p (A(1-x) - x^{1-p} (1 + Bx^p)) = 0,$$

$$\Rightarrow x^p (A - Ax - x^{1-p} - B) = 0.$$ 

Letting $$g(x) = A - Ax - x^{1-p} - B$$ it follows that:

$$-g(0) = A - B,$$

$$\quad \lim_{x \to \infty} g(x) = -\infty,$$

$$-g'(x) = -A - (1-p)x^{p-1} < 0.$$
Consequently, we have two cases:

\[ A > B \]
\[ A < B \]

Therefore, the threshold condition for an endemic equilibrium is that

\[
\frac{AN_p}{1 + Bx^p} < \frac{Cp^{p-1}}{x} \\
\Rightarrow \frac{BN^*}{\alpha} > 1
\]

\[ \alpha \sigma_p \]

(e) Starting from the fixed points we have:

\[
A (1-x) x^p - x = 0 \\
\Rightarrow A (1-x) x^p - x (1 + Bx^p) = 0 \\
\Rightarrow x (A (1-x) x^{p-1} - (A+B) x^p - 1) = 0 \\
\Rightarrow x \left( \frac{A}{x^p} - (A+B) x^p - 1 \right) = 0
\]

Letting \( g(x) = A x^{p-1} - (A+B) x^p - 1 \) it follows that

\[
- \lim_{x \to \infty} g(x) = -\infty \\
\Rightarrow - g'(x) = 0 \\
p' = \frac{(p-1) A x^{p-2} - p (A+B) x^{p-1}}{x^{p-2} (p-1) A x^2 (A+B)} = 0
\]
\( g(x) \) has a critical point at
\[ x^* = \frac{p-1}{p} A \]

Furthermore,
\[
g'(x^*) = A \left[ \frac{(p-1) A}{p} \left( A+B \right) \right]^{p-1} - \left( A+B \right) \left[ \frac{(p-1) A}{p} \right]^{p-1} = A \left( \frac{p-1}{p} \right)^{p-1} \frac{A^{p-1}}{(A+B)^{p-1}} - \left( \frac{p-1}{p} \right)^{p} \frac{A^{p}}{(A+B)^{p-1}}
\]

\[
= A^p \left( p-1 \right)^{p-1} \frac{p - (p-1)^p A^p - p^p (A+B)^{p-1}}{p^p (A+B)^{p-1}}
\]

Consequently, \( g'(x^*) > 0 \) implies
\[
A^p \left( p-1 \right)^{p-1} \frac{p - (p-1)^p A^p - p^p (A+B)^{p-1}}{p^p (A+B)^{p-1}} > 0
\]
is the threshold condition.
\[
\Rightarrow A^p \left( p-1 \right)^{p-1} p > (p-1)^p A^p + p^p (A+B)^{p-1}
\]

(f) If \( p = 1 \) and \( p > 0 \), it follows that
\[
\lim_{t \to \infty} \frac{R(t)}{1 - (R(t))^q} = \infty
\]
That is the infection rate grows with infections which is contrary to the fact that with more infections there are less susceptibles to infect.
In cases of constant recruitment, the limiting system and the original system usually have the same qualitative dynamics. Consider the following SIV model with constant recruitment and vaccination:

\[ \dot{S} = \Delta - \beta S I - \nu S \]
\[ \dot{I} = \beta S I - (\nu + \gamma) I \]
\[ \dot{V} = \gamma I - \nu V. \]

(a) What are the units of \( \Delta, \beta, \nu, \) and \( \gamma \)?

(b) Interpret in practical terms what the constants represent physically.

(c) Find an equation for \( N \) and solve this equation.

(d) Show that \( \hat{N} + N(t) = \Delta/\nu \)

**Solution:**

(a) \([\Delta] = \text{pop/time}\)
\([\beta] = \text{1/time}\)
\([\nu] = \text{1/time}\)
\([\gamma] = \text{1/time}\). 

(b) \( \Delta \) is a constant per capita birth rate, \( \beta \) is an infection rate, \( \nu \) is a death rate, and \( \gamma \) is a recovery rate.

(c) \( \dot{N} = \dot{S} + \dot{I} + \dot{V} = \Delta - \nu N. \) If we consider the homogeneous equation:

\[ N_{\text{H}} + \nu N_{\text{H}} = 0 \]

it follows that the solution to the homogeneous equation is:

\[ N_{\text{H}} = c \exp(-\nu t). \]
A particular solution to the homogeneous equation satisfies
\[ N_p + v N_p = \Delta \]
\[ \Rightarrow N_p = \frac{\Delta}{v}. \]
Therefore, the general solution is given by:
\[ N(t) = \frac{\Delta}{v} + c \exp(-vt). \]

(c). Clearly,
\[ \lim_{t \to \infty} N(t) = \lim_{t \to \infty} \left( \frac{\Delta}{v} + c \exp(-vt) \right) = \frac{\Delta}{v}. \]

#5.
A disease is introduced by two visitors into a town with 1200 inhabitants. An average infective is in contact with 4 inhabitants per day. The average duration of the infective period is 6 days, and recovered infectives are immune against reinfection. Assuming an SIR model, estimate how many inhabitants would have to be immunized to avoid an epidemic.

Solution:
Assuming a completely susceptible population it follows that
\[ R_0 = \# \text{ of secondary infections} \]
\[ = 4 \times 6 \]
\[ = 24. \]
Moreover,
\[ R_0 = 2.4 = \frac{\beta N}{\alpha} \]
\[ \Rightarrow \beta = \frac{2.4}{1200.6} = \frac{1}{1200} = 1.10^{-3}. \]
Consequently, if we want to reduce the reproduction number below one, we need

\[ \frac{\beta S_0}{k} < 1 \]

\[ \Rightarrow S_0 < \frac{k}{\beta} \]

\[ \Rightarrow S_0 < \frac{1}{6 \cdot 3 \cdot 10^3} \]

\[ = 500 \]

Therefore, 700 people need to be immunized to guarantee the infection does not spread.