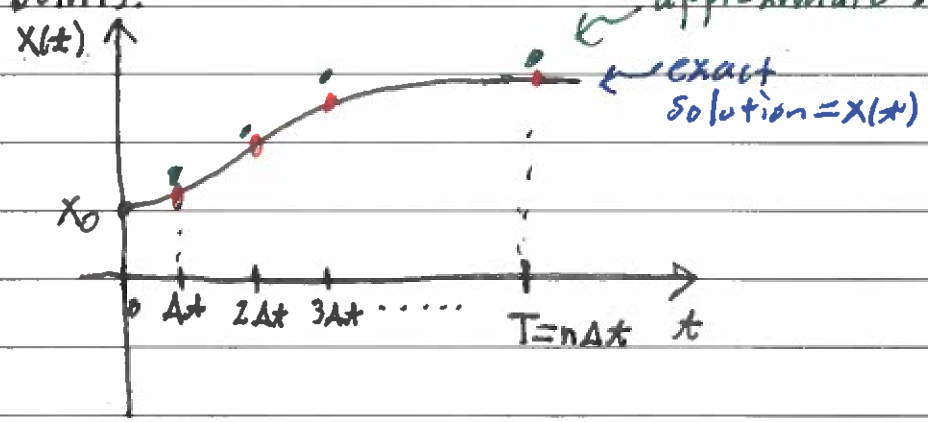
Numerical Solutions to ODEs

How can we appoximate solutions to

$$\dot{x} = f(x)$$
$$x(0) = x_0$$

on a computer?

Challenge: Computers can only store discrete variables.
The key idea is to approximate the solution at n discrete points.



Euler's Method:

How do we approximate $x(\Delta t)$?

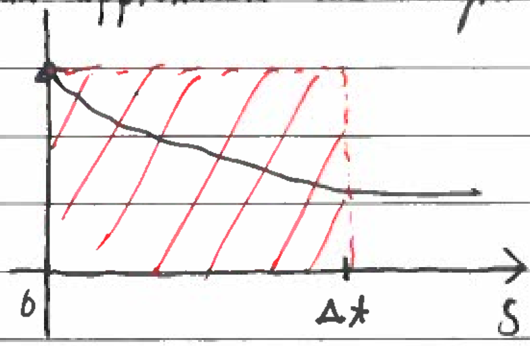
$$\dot{x} = f(x)$$

$$\Rightarrow \int_0^{\Delta t} \frac{dx}{ds} = \int_0^{\Delta t} f(x(s))\,ds$$

$$\Rightarrow x(\Delta t) - x(0) = \int_0^{\Delta t} f(x(s))\,ds$$

$$\Rightarrow \boxed{x(\Delta t) = x_0 + \int_0^{\Delta t} f(x(s))\,ds}$$

We can approximate the integral by the left hand rule:



$$\Rightarrow \int_0^{\Delta t} f(x(s))\,ds \approx f(x_0)\Delta t$$

If we let $\tilde{x}_1$: denote the approximation at $t=\Delta t$ we have

$$x(\Delta t) = x_0 + \int_0^{\Delta t} f(x(s))\,ds$$
$$\Rightarrow \quad \tilde{x}_1 = x_0 + f(x_0)\Delta t$$

We can do a similar approach to get $\tilde{x}_2$:
$$\tilde{x}_2 = \tilde{x}_1 + f(\tilde{x}_1)\Delta t$$

Generalizing, we get <u>Euler's Method</u>:

$$\tilde{x}_0 = x_0$$
$$\tilde{x}_{i+1} = \tilde{x}_i + f(\tilde{x}_i)\Delta t$$

<u>How Accurate:</u>

What is the diffence between $x(i\Delta t)$ and $\tilde{x}_i$? That is, can we estimate $|x(\Delta t) - \tilde{x}_1|$?

Taylor Expanding we have:
$$x(\Delta t) = x(0) + \dot{x}(0)\Delta t + \tfrac{1}{2}\ddot{x}(c)\Delta t^2, \quad c \in [0, \Delta t].$$
$$= x_0 + f'(x_0)\Delta t + \tfrac{1}{2} f(x(c)) f'(x(c))\Delta t^2$$
$$= \tilde{x}_1 + \tfrac{1}{2} f(x(c)) f'(x(c))\Delta t^2$$

Therefore,

$$|x(\Delta t) - x_1| \approx \tfrac{1}{2} f(x(c)) f'(x(c)) \Delta t^2$$

$$\Rightarrow |x(\Delta t) - x_1| \leq M \Delta t^2$$

Definition of "Big-Oh" notation.

$$\Rightarrow \underbrace{|x(\Delta t) - x_1| = \mathcal{O}(\Delta t^2)}_{\text{local truncation error, goes to zero as } \Delta t \to 0.}$$

More generally,

$$x(t+\Delta t) = x(t) + \dot{x}(t)\Delta t + \tfrac{1}{2}\ddot{x}(c)\Delta t^2$$

$$\Rightarrow x(t+\Delta t) = x(t) + f(x(t))\Delta t + \tfrac{1}{2} f(x(c)) f'(x(c)) \Delta t^2$$

$$\Rightarrow |x(i\Delta t) - \tilde{x}_i| = \mathcal{O}(\Delta t^2)$$

The global error is the sum of all the local truncation errors:

$$\therefore \text{Global error} = \sum_{i=1}^{n} |x(i\Delta t) - \tilde{x}_i| = n\,\mathcal{O}(\Delta t^2),$$

However, $n\Delta t = T \Rightarrow n = T/\Delta t$

$$\Rightarrow \boxed{\text{Global Error} = \mathcal{O}(\Delta t)}$$

Improved Euler's Method:

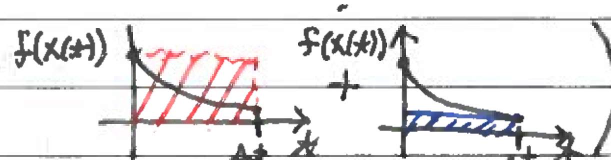
If we go back to our approximation we have

$$\dot{x} = f(x)$$

$$\Rightarrow x(\Delta t) = x_0 + \int_0^{\Delta t} f(x(s))\,ds$$

We now approximate the integral by the average of the left and right hand rules

$$\int_0^{\Delta t} f(x(t))\,dt \approx \frac{1}{2}\Bigg(\qquad + \qquad\Bigg)$$



$$\approx \frac{1}{2} f(x_0) + \frac{1}{2} f(x(\Delta t))$$

However, we can approximate $x(\Delta t)$ using Euler's method

$$x(\Delta t) \approx x_0 + f(x_0)\Delta t$$

Therefore, the first approximate is given by

$$\tilde{X}_1 = x_0 + \frac{1}{2}\Big(f(x_0) + f(x_0 + f(x_0)\Delta t)\Big)\Delta t$$

Generalizing we get the modified or improved Euler's method:

$$\boxed{\begin{aligned} &\tilde{X}_0 = x_0 \\ &\hat{X}_{i+1} = \tilde{X}_i + f(\tilde{X}_i)\Delta t \\ &\tilde{X}_{i+1} = \tilde{X}_i + \tfrac{1}{2}\big(f(\tilde{X}_i) + f(\hat{X}_{i+1})\big)\Delta t \end{aligned}}$$

How Accurate?

Taylor expanding we have

$$\begin{aligned} \tilde{X}_1 &= x_0 + \tfrac{1}{2}\big(f(x_0) + f(x_0 + f(x_0)\Delta t)\big)\Delta t \\ &= x_0 + \tfrac{1}{2}f(x_0)\Delta t + \tfrac{1}{2}\big(f(x_0) + f'(x_0)f(x_0)\Delta t + \overset{\nearrow \text{Some constant}}{C}\Delta t^2\big)\Delta t \\ &= x_0 + f(x_0)\Delta t + \tfrac{1}{2}f'(x_0)f(x_0)\Delta t^2 + C\Delta t^3 \end{aligned}$$

We also have that

$$x(\Delta t) = x(0) + \dot{x}(0)\Delta t + \tfrac{1}{2}\ddot{x}(0)\Delta t^2 + \tfrac{1}{6}\dddot{x}(c)\Delta t^3$$

Therefore,

$$|\tilde{X}_1 - x(\Delta t)| = \mathcal{O}(\Delta t^3) \leftarrow \text{Local truncation error}$$

The global error is therefore

$$\text{global error} = \mathcal{O}(\Delta t^2)$$