MTH 351/651
Fall 2025
Exam 1
09/26/25

Name (Print): Key

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	15	
2	10	
3	10	
4	20	
5	25	
6	20	
Total:	100	

Do not write in the table to the right.

1. (15 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

C **I** Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function and consider the dynamical system $\dot{x} = f(x)$. If x^* satisfies $f(x^*) = 0$ and $f'(x^*)=0$ then x^* is a semi-stable fixed point.

C **I** Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function and consider the dynamical system $\dot{x} = f(x)$. If x^* satisfies $f(x^*) = 0$ and x^* is unstable then $f'(x^*) > 0$.

C **I** Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function and consider the dynamical system $\dot{x} = f(x)$. If x^* satisfies $f(x^*) = 0$ then $\ddot{x}\big|_{x=x^*} = 0$.

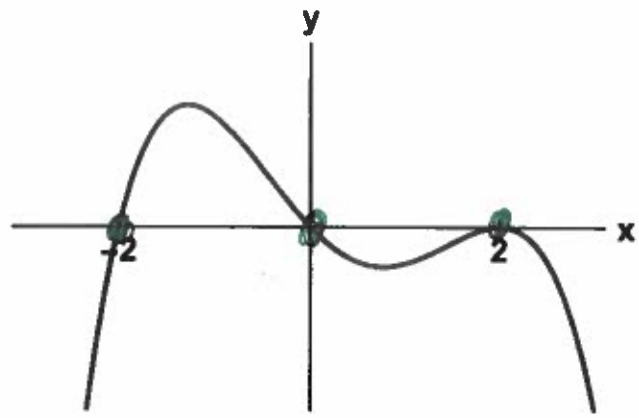
C **I** Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function and consider the dynamical system $\dot{x} = f(x)$. If this system has two fixed points then both of these points can be stable.

C **I** Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function and consider the dynamical system $\dot{x} = f(x)$ If this system as one fixed point x^* and it is stable and $x(t)$ solves $\dot{x} = f(x)$ with initial condition $x(0) = x_0$, then there exists a T satisfying $0 < T < \infty$ such that $x(T) = x^*$.

2. (10 points) Consider the differential equation

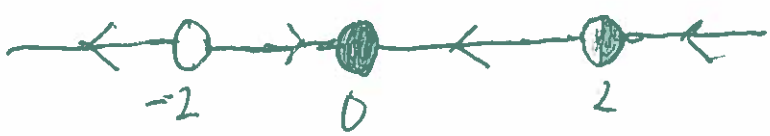
$$\dot{x} = f(x),$$

where $f(x)$ is plotted below.

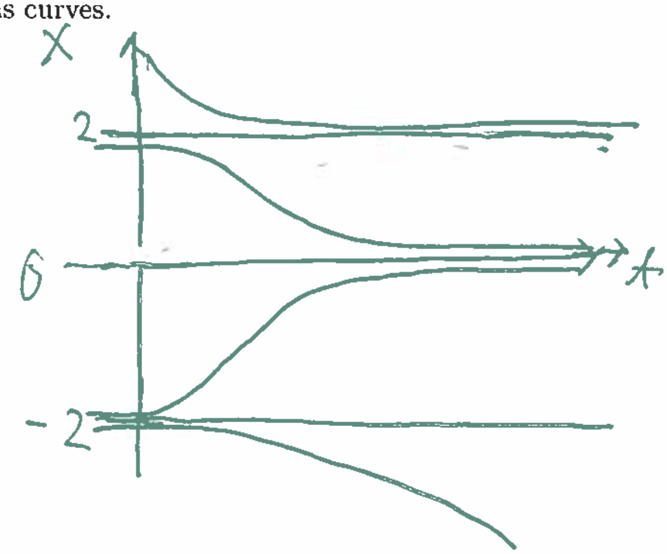


(a) (2 points) **Short Answer:** On the figure indicate any fixed points, i.e., equilibrium points, for this differential equation.

(b) (3 points) **Short Answer:** Sketch a one-dimensional phase portrait for this problem.



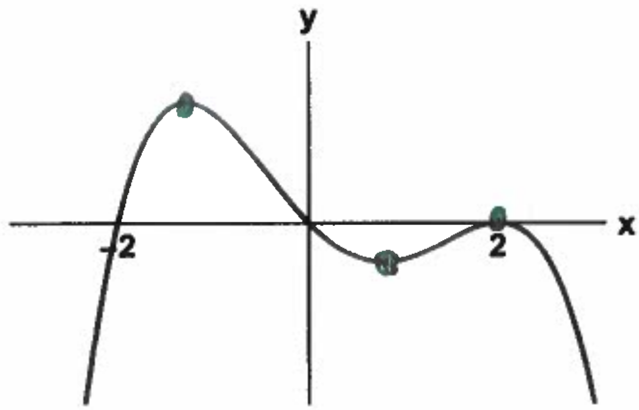
(c) (5 points) **Short Answer:** On one axis, sketch the corresponding solution curves $x(t)$ for this problem. Your solution curves should contain all possible qualitatively different types of solutions curves.



3. (10 points) Consider the differential equation

$$\dot{x} = -\frac{dV}{dx},$$

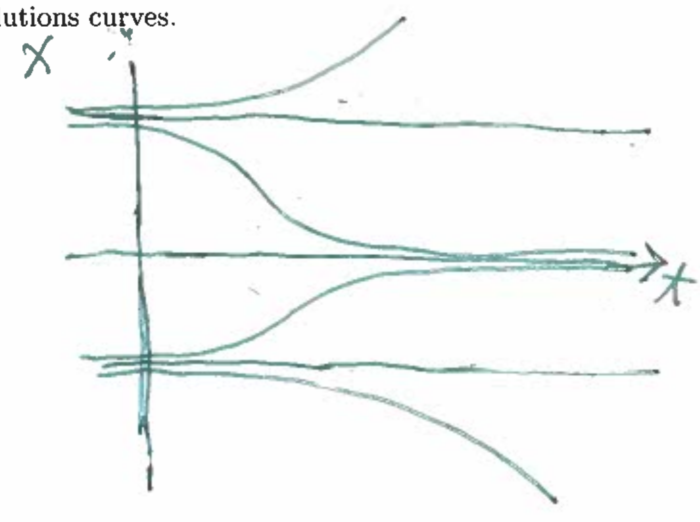
where $V(x)$ is plotted below.



(a) (2 points) **Short Answer:** On the figure indicate any fixed points, i.e., equilibrium points, for this differential equation.

(b) (3 points) **Shoart Answer:** Sketch a one-dimensional phase portrait for this problem.

(c) (5 points) **Short Answer:** On one axis, sketch the corresponding solution curves $x(t)$ for this problem. Your solution curves should contain all possible qualitatively different types of solutions curves.



4. (20 points) One model for spread of malaria assumes the population of mosquitoes is in equilibrium and models the dynamics of the infected population I by the differential equation

$$\dot{I} = \frac{\alpha I}{A+I}\left(1 - \frac{I}{N}\right) - \mu I,$$

where $\alpha, A, N, \mu > 0$ are all constants.

(a) (5 points) Determine the dimensions of the constants α, A, N, μ.

$[\alpha] = \frac{\text{pop}}{\text{time}}$ $\quad [N] = \text{pop}$

$[A] = \text{pop}$ $\quad [\mu] = \frac{1}{\text{time}}$

(b) (5 points) By making an appropriate change of variables, show that this system can be put into the following dimensionless form

$$\frac{dx}{d\tau} = f(x) = \frac{ax}{b+x}(1-x) - x,$$

where a and b are quantities to be determined.

Let $x = I/N$, $\tau = \mu t$. Therefore,

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \mu\frac{d}{d\tau}$$

Consequently,

$$\dot{I} = \mu\frac{dI}{d\tau} = \mu N\frac{dx}{d\tau} = \frac{\alpha N x}{A+Nx}(1-x) - \mu N x$$

$$\Rightarrow \frac{dx}{d\tau} = \frac{\alpha\, x}{N\mu(A/N + x)}(1-x) - x = \frac{ax}{b+x}(1-x) - x.$$

(c) (5 points) For the dimensionless system derived in part (b), for what values of a and b is the fixed point $x^* = 0$ stable.

Letting $f(x) = \frac{ax(1-x)}{b+x} - x$ we have that

$$f'(0) = \frac{a}{b} - 1$$

and thus $x^* = 0$ is stable if

$$\frac{a}{b} < 1,$$

i.e., $a < b$.

5. (25 points) For $\mu \in \mathbb{R}$, consider the following dynamical system:

$$\dot{x} = x(\mu - 3) - x^2.$$

(a) (5 points) Determine the fixed points for this problem.

$x = 0$ and $\mu - 3 - x = 0$

$\Rightarrow x = 0$ and $x = \mu - 3$.

(b) (5 points) Classify the stability of the fixed points for this problem as a function of μ. You do not have to worry about the case when the fixed points are semistable.

Letting $f(x) = x(\mu - 3) - x^2$ we have $\lim_{x \to \infty} f(x) = -\infty$.

Consequently, we have two cases

<u>Case 1 ($\mu < 3$):</u>

$x = 0$ stable

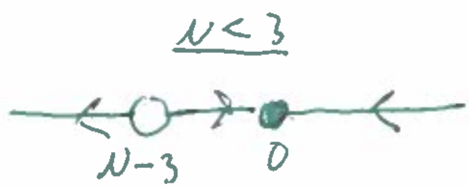
$x = \mu - 3$ unstable

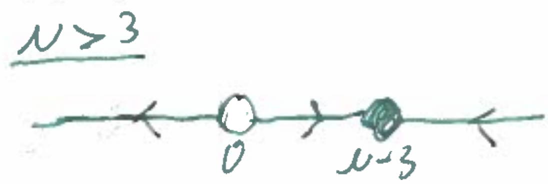
<u>Case 2 ($\mu > 3$):</u>

$x = 0$ unstable

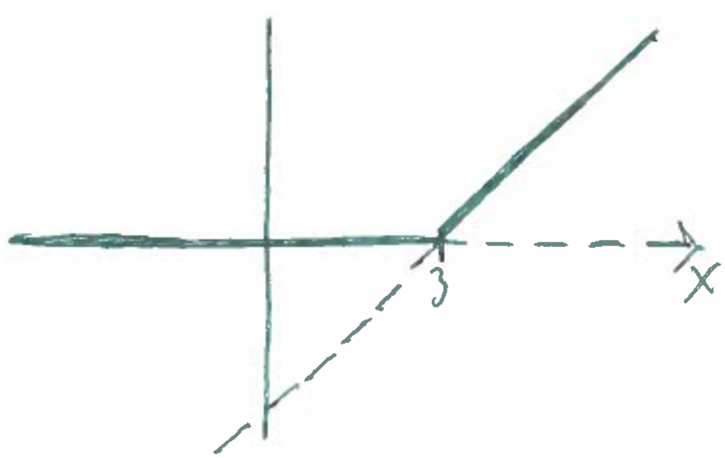
$x = \mu - 3$ stable.

(c) (5 points) Sketch all qualitatively different phase portraits that occur as μ is varied. You do not have to worry about the case when the fixed points are semistable.

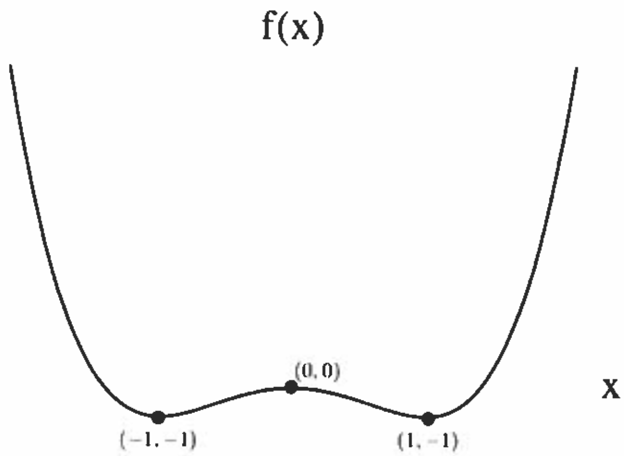




(d) (10 points) Sketch a bifurcation diagram for this problem as a function of μ and identify the bifurcations that occur.



6. (20 points) Consider the following differential equation $\dot{x} = \mu + f(x)$ where $\mu \in \mathbb{R}$ is a constant and $f(x)$ is plotted below.



(a) (1 point) **Short Answer:** For all values of μ, what is $\lim_{x\to\infty} \mu + f(x)$?

∞

(b) (3 points) **Short Answer:** For what values of μ does this system have no fixed points?

$\mu > 1$

(c) (3 points) **Short Answer:** For what values of μ does this system have two fixed points?

$\mu < 0$

(d) (3 points) **Short Answer:** For what values of μ does this system have four fixed points?

$0 < \mu < 1$

(e) (10 points) **Short Answer:** Sketch a bifurcation diagram for this problem. On your diagram be sure to label and classify any bifurcation points.