MTH 351/651
Fall 2025
Exam 2
10/31/25

Name (Print): Key

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	15	
2	10	
3	15	
4	25	
5	25	
6	10	
Total:	100	

Do not write in the table to the right.

1. (15 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

   C **I** Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth, periodic function, and consider the dynamical system $\dot{\theta} = f(\theta)$ defined on the unit circle. If this system has exactly one fixed point θ^* on the unit circle, then θ^* cannot be asymptotically stable.

   C **I** Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth, periodic function, and consider the dynamical system $\dot{\theta} = f(\theta)$ defined on the unit circle. If $f(\theta) > 0$ then on the unit circle solutions $\theta(t)$ to this equation are periodic in time.

   C **I** Let f and g be smooth functions. If the system $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ has exactly one fixed point at the origin $(0, 0)$ and the eigenvalues of the Jacobian matrix at $(0, 0)$ are $\lambda_{1,2} = -1 \pm i$ then all solution curves $(x(t), y(t))$ satisfy $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$.

   C **I** Let f and g be smooth functions. If the system $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ has exactly one fixed point at the origin $(0, 0)$ and the eigenvalues of the Jacobian matrix at $(0, 0)$ are $\lambda_{1,2} = i$ then this system has nonlinear centers around the origin.

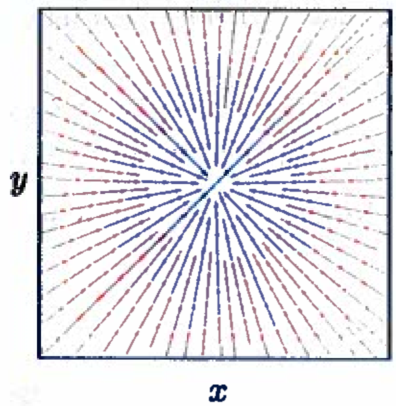
   C **I** Let f and g be smooth functions. If the system $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ has exactly one fixed point at the origin $(0, 0)$ and the eigenvalues of the Jacobian matrix at $(0, 0)$ are $\lambda_{1,2} = \pm 1$ then the origin is unstable.

2. (10 points) **Short Answer:** Consider the following linear system of differential equations

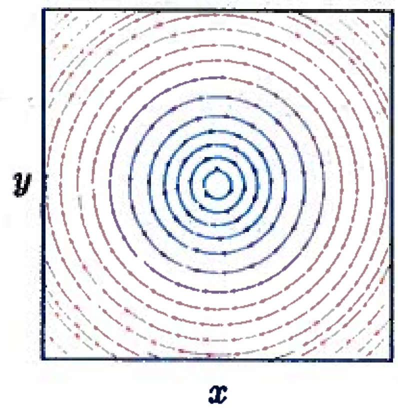
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}.$$

Match the following possible phase portraits for this system with the corresponding matrices listed below. Note, there are more matrices than phase portraits, cross out any matrices that are not used.

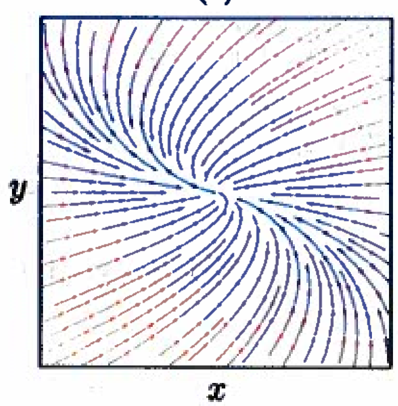
(a)



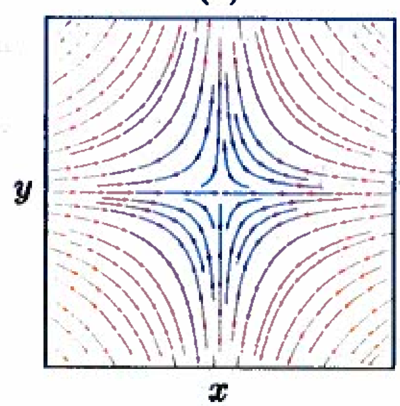
(b)



(c)



(d)



(b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\cancel{A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$, (d) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\cancel{A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$,

$\cancel{A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$, (c) $A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$, (a) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\cancel{A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$.

3. (15 points) The following linear system of differential equations describes the dynamics for the romantic love that an individual Romeo feels for another individual Juliet, denoted R, and the romantic love that Juliet feels for Romeo, denoted J:

$$\dot{R} = J,$$
$$\dot{J} = -R + J.$$

In this problem negative values of J and R indicate hatred for the other individual.

(a) (5 points) **Short Answer:** Characterize the romantic style of Romeo and Juliet. That is, briefly describe how Romeo and Juliet respond to each others romantic feelings as well as their own.

Romeo responds positively to Juliets affection. Juliet responds negatively to Romeo's affection but she reinforces affection for him the more she has feelings for him.

(b) (5 points) Classify the fixed point at the origin.

$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$\lambda_1 + \lambda_2 = 1$

$\lambda_1 \lambda_2 = 1$

$\Rightarrow \lambda_1 = 1 - \lambda_2$

$(1-\lambda_2)\lambda_2 = 1$

$\Rightarrow \lambda_2^2 - \lambda_2 + 1 = 0$

$\Rightarrow \lambda_2 = \frac{1 \pm \sqrt{1-4}}{2}$

$\Rightarrow$ $(0,0)$ is an unstable spiral

(c) (5 points) **Short Answer:** Based on your classification of the fixed point, what does your result imply about their romantic affair?

The feelings for each other oscillates between love and hate and both emotions grow without bound.

4. (25 points) Consider the following differential equation **defined on the circle**:

$$\dot{\theta} = \mu - \cos(\theta),$$

where $\mu \in \mathbb{R}$ is a parameter. For this problem, by unit circle I am assuming the convention that $\theta \in [-\pi, \pi]$.

(a) (10 points) If $\mu = 0$, determine the fixed points of this system and analyze their stability.

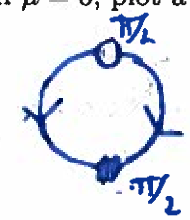
Let $f(\theta) = -\cos\theta$. The fixed points are $\theta = \pm \pi/2$.
Differentiating we have

$f'(\theta) = \sin\theta$

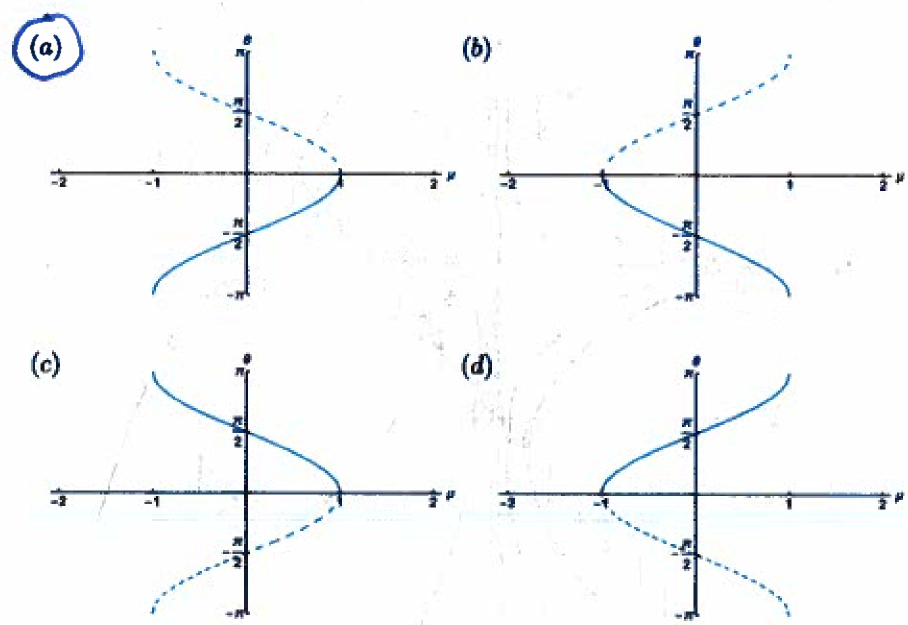
$f'(\pi/2) = 1 \Rightarrow \pi/2$ is unstable,

$f'(-\pi/2) = -1 \Rightarrow -\pi/2$ is stable.

(b) (5 points) If $\mu = 0$, plot a phase portrait for this system on the unit circle.



(c) (5 points) **Short Answer:** Circle the figure below which corresponds to the bifurcation diagram for this system.



(d) (5 points) **Short Answer:** Based on your answer above, list the bifurcation points for this problem and and classify each bifurcation point.

Bifurcation points at $\mu = -1$ and 1 are both saddle node bifurcations.

5. (25 points) Consider the following system of differential equations:

$$\dot{x} = -(x^2 - xy),$$
$$\dot{y} = 1 - y.$$

(a) (15 points) Calculate the nullclines and fixed points for this system and analyze the local stability of the fixed points.

$\underline{\dot{x}=0}$

$x=0, y=x$

$\underline{\dot{y}=0}$

$y=1$

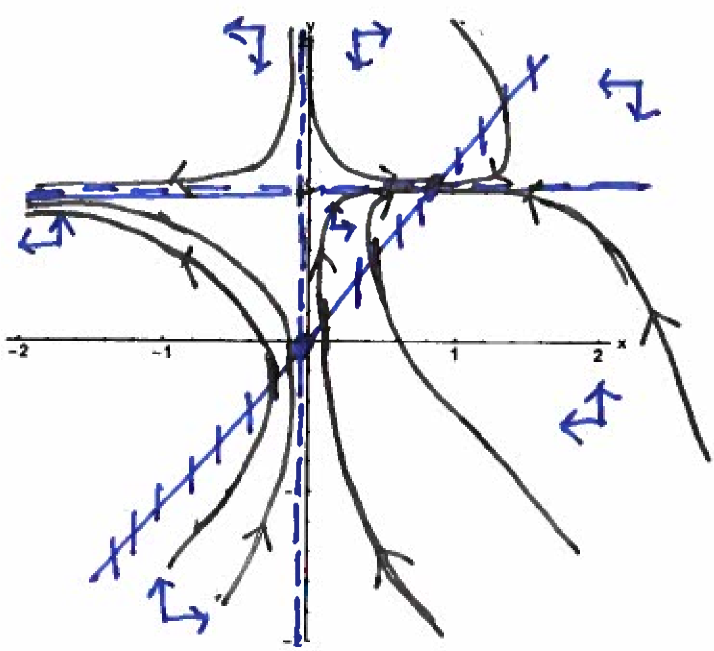
Fixed points at $(0,1)$ and $(1,1)$

$$J(x,y)=\begin{bmatrix}-2x+y & y\\ 0 & -1\end{bmatrix}$$

$$J(0,1)=\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix} \Rightarrow (0,1) \text{ is a saddle}$$

$$J(1,1)=\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix} \Rightarrow (1,1) \text{ is a stable node.}$$

(b) (10 points) On the axes below, sketch the nullclines and a reasonable phase portrait for this system.



6. (10 points) Consider the following system of differential equations:

$$\dot{x} = -y - x^3 - xy^2,$$
$$\dot{y} = x - x^2y - y^3.$$

For all solutions $(x(t), y(t))$ to this system of equations, show that the radial coordinate defined by $r^2 = x^2 + y^2$ satisfies

$$\lim_{t\to\infty} r(t) = 0.$$

$$r^2 = x^2 + y^2$$
$$\Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$$
$$= (-y - x^3 - xy^2)x + (x - x^2y - y^3)y$$
$$= -xy - x^4 - x^2y^2 + xy - x^2y^2 - y^4$$
$$= -2x^2y^2 - x^4 - y^4$$
$$= -(x^4 + 2x^2y^2 + y^4)$$
$$= -(x^2+y^2)^2$$
$$= -r^4$$
$$\Rightarrow \dot{r} = -r^3.$$

This is a 1-d system with the only fixed point $r=0$ with phase portrait

Therefore,

$$\lim_{t\to\infty} r(t) = 0$$