

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a “rough draft”; your submission with well-organized calculations and relevant explanations should be thought of as your “final draft”.

Beyond the Poincaré–Bendixson Theorem, Jacobian analysis offers another way to determine whether a limit cycle exists, as summarized in the following theorem.

Theorem 1: Hopf-Bifurcation Theorem

Consider the planar system $\dot{x} = f(x, y; \mu)$ and $\dot{y} = g(x, y; \mu)$, where $\mu \in \mathbb{R}$ is a parameter and f, g are smooth functions. Suppose the system admits a family of fixed points $(x^*(\mu), y^*(\mu))$ depending smoothly on μ and let $\lambda_1(\mu)$ and $\lambda_2(\mu)$ denote the eigenvalues of the Jacobian matrix evaluated at $(x^*(\mu), y^*(\mu))$. If there exists a value μ^* such that

1. $\operatorname{Re}(\lambda_1(\mu^*)) = \operatorname{Re}(\lambda_2(\mu^*)) = 0$,
2. $\frac{d}{d\mu} [\operatorname{Re}(\lambda_j(\mu))] \big|_{\mu=\mu^*} > 0$ for $j = 1, 2$,

then there exists $\varepsilon > 0$ such that

- for $\mu \in (\mu^* - \varepsilon, \mu^*)$, $(x^*(\mu), y^*(\mu))$ is a stable spiral,
- for $\mu \in (\mu^*, \mu^* + \varepsilon)$, $(x^*(\mu), y^*(\mu))$ is an unstable spiral,
- for $\mu \in (\mu^* - \varepsilon, \mu^*)$ or $\mu \in (\mu^*, \mu^* + \varepsilon)$ there exists a limit cycle surrounding $(x^*(\mu), y^*(\mu))$.

This theorem essentially states that as the parameter μ is varied, if a stable spiral becomes an unstable spiral at $\mu = \mu^*$, then a stable limit cycle emerges or an unstable limit cycle disappears at $\mu = \mu^*$. The point $\mu = \mu^*$ is called a **Hopf bifurcation**. **However, the theorem does not tell us whether the limit cycle exists for $\mu < \mu^*$ or for $\mu > \mu^*$** ; that is, the theorem alone cannot determine whether an unstable limit cycle disappears or a stable one emerges at the bifurcation point. If a stable limit cycle is born, we say the system undergoes a **supercritical Hopf bifurcation** while if an unstable limit cycle disappears we say the system undergoes a **subcritical Hopf bifurcation**.

Problems to be completed by all students

Problem 1. Rewrite the Hopf-bifurcation theorem if condition number 2 is replaced with

$$\frac{d}{d\mu} [\operatorname{Re}(\lambda_j(\mu))] \big|_{\mu=\mu^*} < 0 \text{ for } j = 1, 2.$$

Note: The definition of sub and supercritical Hopf bifurcation remains the same in this case in the sense that the prefix *super* is used when there exists a stable limit cycle while the prefix *sub* is used when there exists an unstable limit cycle.

Problem 2. For each of the following systems, show that a Hopf bifurcation occurs at the origin for some value of μ and then use Mathematica or some other software to plot the phase portrait near the Hopf bifurcation point to determine whether the bifurcation is subcritical or supercritical.

- (a) $\dot{x} = y + \mu x$ and $\dot{y} = -x + \mu y - x^2 y$,
- (b) $\dot{x} = \mu x + y - x^3$ and $\dot{y} = -x + \mu y - 2y^3$,
- (c) $\dot{x} = \mu x + y - x^2$ and $\dot{y} = -x + \mu y - 2x^3$.

Problem 3. Consider the following dimensionless predator prey model

$$\dot{x} = x^2(1 - x) - xy, \dot{y} = -ay + xy,$$

where $x, y \geq 0$ and $a > 0$ is a parameter.

- (a) Identify which species is the prey and which is the predator and provide biological interpretations of each term in the model. In particular, try to give a reasonable explanation for the term $x^2(1 - x)$.
- (b) Find the fixed points for this system and use the Jacobian to classify them as a function of a .
- (c) For the case $a > 1$ sketch the nullclines, indicate the directions of the flow in the region partitioned by the nullclines, and sketch the phase portrait for this system.
- (d) Show that a Hopf bifurcation occurs when $a = \frac{1}{2}$.
- (e) Sketch all qualitatively different phase portraits, except for weird edge cases, that can occur for $0 < a < 1$ and use this to determine if you think the Hopf bifurcation was sub or supercritical.