Homework #10
MTH 351/651: Fall 2025
Instructor: John Gemmer

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a "rough draft"; your submission with well-organized calculations and relevant explanations should be thought of as your "final draft".

Beyond the Poincaré–Bendixson Theorem, Jacobian analysis offers another way to determine whether a limit cycle exists, as summarized in the following theorem.

Theorem 1: Hopf-Bifurcation Theorem

Consider the planar system $\dot{x} = f(x, y; \mu)$ and $\dot{y} = g(x, y; \mu)$, where $\mu \in \mathbb{R}$ is a parameter and f, g are smooth functions. Suppose the system admits a family of fixed points $(x^*(\mu), y^*(\mu))$ depending smoothly on μ and let $\lambda_1(\mu)$ and $\lambda_2(\mu)$ denote the eigenvalues of the Jacobian matrix evaluated at $(x^*(\mu), y^*(\mu))$. If there exists a value μ^* such that

1. $\text{Re}(\lambda_1(\mu^*)) = \text{Re}(\lambda_2(\mu^*)) = 0$,
2. $\frac{d}{d\mu}\left[\text{Re}(\lambda_j(\mu))\right]\big|_{\mu=\mu^*} > 0$ for $j = 1, 2$,

then there exists $\varepsilon > 0$ such that

- for $\mu \in (\mu^* - \varepsilon, \mu^*)$, $(x^*(\mu), y^*(\mu))$ is a stable spiral,
- for $\mu \in (\mu^*, \mu^* + \varepsilon)$, $(x^*(\mu), y^*(\mu))$ is an unstable spiral,
- for $\mu \in (\mu^* - \varepsilon, \mu^*)$ or $\mu \in (\mu^*, \mu^* + \varepsilon)$ there exists a limit cycle surrounding $(x^*(\mu), y^*(\mu))$.

This theorem essentially states that as the parameter μ is varied, if a stable spiral becomes an unstable spiral at $\mu = \mu^*$, then a stable limit cycle emerges or an unstable limit cycle disappears at $\mu = \mu^*$. The point $\mu = \mu^*$ is called a **Hopf bifurcation. However, the theorem does not tell us whether the limit cycle exists for $\mu < \mu^*$ or for $\mu > \mu^*$**; that is, the theorem alone cannot determine whether an unstable limit cycle disappears or a stable one emerges at the bifurcation point. If a stable limit cycle is born, we say the system undergoes a **supercritical Hopf bifurcation** while if an unstable limit cycle disappears we say the system undergoes a **subcritical Hopf bifurcation**.

Problems to be completed by all students

Problem 1. Rewrite the Hopf-bifurcation theorem if condition number 2 is replaced with

$$\frac{d}{d\mu}\left[\text{Re}(\lambda_j(\mu))\right]\big|_{\mu=\mu^*} < 0 \text{ for } j = 1, 2.$$

Note: The definition of sub and supercritical Hopf bifurcation remains the same in this case in the sense that the prefix *super* is used when there exists a stable limit cycle while the prefix *sub* is used when there exists an unstable limit cycle.

Problem 2. For each of the following systems, show that a Hopf bifurcation occurs at the origin for some value of μ and then use Mathematica or some other software to plot the phase portrait near the Hopf bifurcation point to determine whether the bifurcation is subcritical or supercritical.

(a) $\dot{x} = y + \mu x$ and $\dot{y} = -x + \mu y - x^2 y$,

(b) $\dot{x} = \mu x + y - x^3$ and $\dot{y} = -x + \mu y - 2y^3$,

(c) $x = \mu x + y - x^2$ and $\dot{y} = -x + \mu y - 2x^3$.

Problem 3. Consider the following dimensionless predator prey model

$$\dot{x} = x^2(1-x) - xy, \dot{y} \quad = -ay + xy,$$

where $x, y \geq 0$ and $a > 0$ is a parameter.

(a) Identify which species is the prey and which is the predator and provide biological interpretations of each term in the model. In particular, try to give a reasonable explanation for the term $x^2(1-x)$.

(b) Find the fixed points for this system and use the Jacobian to classify them as a function of a.

(c) For the case $a > 1$ sketch the nullclines, indicate the directions of the flow in the region partitioned by the nullclines, and sketch the phase portrait for this system.

(d) Show that a Hope bifurcation occurs when $a = \frac{1}{2}$.

(e) Sketch all qualitatively different phase portraits, except for weird edge cases, that can occur for $0 < a < 1$ and use this to determine if you think the Hopf bifurcation was sub or supercritical.

Homework #10

#1.

Solution:

Consider the planar system $\dot{x}=f(x,y;\nu)$, $\dot{y}=g(x,y;\nu)$ where $\nu\in\mathbb{R}$ and f,g are smooth functions. Suppose the system admits a family of fixed points $(x^*(\nu),y^*(\nu))$ depending smoothly on ν and let $\lambda_1(\nu)$ and $\lambda_2(\nu)$ denote the eigenvalues of the Jacobian matrix evaluated at $(x^*(\nu),y^*(\nu))$. If there exists ν^* such that

1. $Re(\lambda_1(\nu^*))=Re(\lambda_2(\nu^*))=0$
2. $\frac{d}{d\nu}Re[\lambda_j(\nu)]\Big|_{\nu=\nu^*}<0$ for $j=1,2$

then there exists $\varepsilon>0$ such that

- for $\nu\in(\nu^*-\varepsilon,\nu^*)$, $(x^*(\nu),y^*(\nu))$ is a unstable spiral.
- for $\nu\in(\nu^*,\nu^*+\varepsilon)$, $(x^*(\nu),y^*(\nu))$ is a stable spiral.
- for $\nu\in(\nu^*-\varepsilon,\nu^*)$ or $\nu\in(\nu^*,\nu^*+\varepsilon)$ there exists a limit cycle surrounding $(x^*(\nu),y^*(\nu))$.

■

#2

(a) $\dot{x}=y+\nu x$
$\dot{y}=-x+\nu y-x^2y$

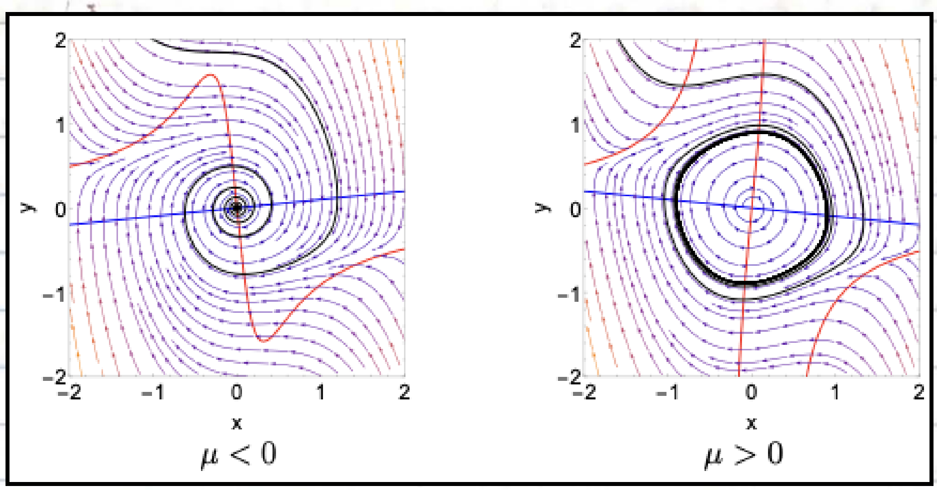
(b) $\dot{x}=\nu x+y-y^3$
$\dot{y}=-x+\nu y-2y^3$

(c) $\dot{x}=\nu x+y-x^2$
$\dot{y}=-x+\nu y-2x^3$

Solution:

(a) $J(0,0) = \begin{bmatrix} \nu & 1 \\ -1 & \nu \end{bmatrix} \Rightarrow \lambda_{1,2} = \nu \pm i$.

Consequently, since $\frac{d}{d\nu}\lambda(\nu)\big|_{\nu=0} > 0$ there is a Hopf-bifurcation.
The phase portraits indicate this is a super-critical Hopf bifurcation:



(b) $J(0,0) = \begin{bmatrix} \nu & 1 \\ -1 & \nu \end{bmatrix} \Rightarrow \lambda_{1,2} = \nu \pm i$

Consequently, since $\frac{d}{d\nu}\lambda(\nu)\big|_{\nu=0} > 0$ there is a Hopf-bifurcation.
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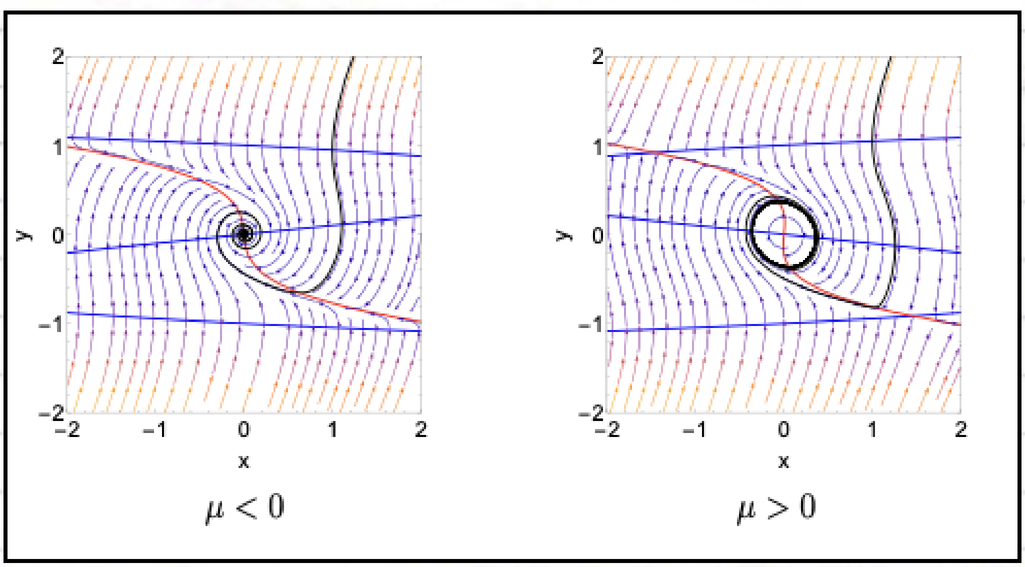


#3

$\dot{x} = x^2(1-x) - xy$

$\dot{y} = -ay + xy$

Solution:

(a) Species x is the prey and y is the predator. The terms have the following interpretation:

- $-ay$ = pure death of predator
- $\pm xy$ = symmetric predation term.
- $x^2(1-x) = x\cdot(1-x)$ = logistic growth with population dependent rate of growth x.

(b) The nullclines satisfy:

$\dot{x}=0$	$\dot{y}=0$
$\cdot\ x=0$	$\cdot\ y=0$
$\cdot\ y=x(1-x)$	$\cdot\ x=a$

The fixed points are therefore

$(0,0);\ (1,0),\ (a, a(1-a))$

Now,

$$J(x,y) = \begin{bmatrix} 2x-3x^2-y & -x \\ y & x-a \end{bmatrix}$$

$$\Rightarrow J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -a \end{bmatrix} \Rightarrow \lambda_{1,2} = 0, -a \Rightarrow (0,0)$$ is not classified by linear analysis.

$$\Rightarrow J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & 1-a \end{bmatrix} \Rightarrow \lambda_1 = -1,\ \lambda_2 = 1-a \Rightarrow (1,0)$$ is a saddle if $a<1$ and is a stable fixed point if $a>1$.

$$\Rightarrow J(a, a(1-a)) = \begin{bmatrix} 2a-3a^2-a+a^2 & -a \\ a(1-a) & 0 \end{bmatrix} = \begin{bmatrix} a(1-2a) & -a \\ a(1-a) & 0 \end{bmatrix}$$

Consequently, the eigenvalues satisfy

$$\lambda_{1,2} = \frac{a(1-2a) \pm \sqrt{a^2(1-2a)^2 - 4a^2(1-a)}}{2}$$

$$= \frac{a(1-2a) \pm a\sqrt{1-4a+4a^2-4+4a}}{2}$$

$$= \frac{a}{2}\left(1-2a \pm \sqrt{4a^2-3}\right).$$

Therefore, we have the following cases:

- If $a < \frac{1}{2}$, $(a, a(1-a))$ is an unstable spiral
- If $\frac{1}{2} < a < \sqrt{3/4}$, $(a, a(1-a))$ is a stable spiral

We can solve for the transition to a saddle by setting

$$1-2a+\sqrt{4a^2-3} = 0$$

$$\Rightarrow \sqrt{4a^2-3} = 2a-1$$

$$\Rightarrow 4a^2-3 = 4a^2-4a+1$$

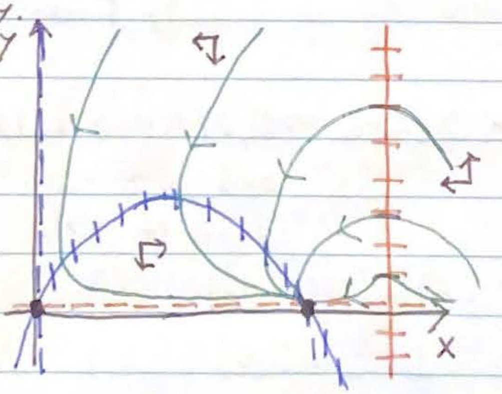
$$\Rightarrow -4 = 4a$$

$$\Rightarrow a = 1.$$

Consequently, we have the additional two cases.

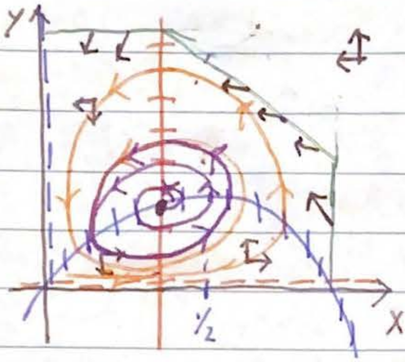
- If $\sqrt{3/4} < a < 1$, $(a, a(1-a))$ is a stable node.
- If $a > 1$, $(a, a(1-a))$ is a saddle but at the same time this fixed point does not exist in the physically relevant regime.

(c) If $a > 1$, the phase portrait in the first quadrant is given by.



(d) Since the stability of the spirals changes when $a=\frac{1}{2}$, a Hopf bifurcation occurs.

(e) If $a<\frac{1}{2}$ we have the following case:



Now, for fixed x, and large y we have

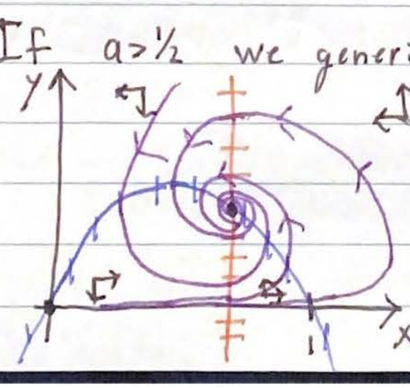
$$\frac{dy}{dx}=\frac{-ay+xy}{x^2(1-x)-xy}\sim\frac{-ay+xy}{-xy}=\frac{-a+x}{x}=\frac{a}{x}-1=g(x).$$

For $a<x<1$, and the fact that $g(a)=0$, $g(1)=a-1$ it follows from the extreme value theorem that for $a<x<1$ and large enough y that

$$a-1\lesssim\frac{dy}{dx}\lesssim 0$$

Consequently, there exists a trapping region as illustrated above. Since the fixed $(a,a(1-a))$ is unstable by Poincare-Bendixon there exists a limit cycle surrounding $(a,a(1-a))$. I tried illustrating outward spirals from $(a,a(1-a))$ in purple and trajectories on the "outside" of the cycle in orange.

If $a>\frac{1}{2}$ we generically have the following case:



*By these two case, I expect the Hopf bifurcation is supercritical.